# TRANSFORMASI Z

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# Hubungan *Recurrence (Sebab Akibat)*

- Kadang-kadang pernyataan *adjacent* dari suatu deret saling berhubungan satu dengan yang lain.
- Sebagai contoh, pernyataan dari suatu deret $\{x_k\} = \{2^k\}$
- bahwa $x_{k+1} = 2^{k+1} = 2 \times 2^k = 2x_k$
- Artinya $x_{k+1} = 2x_k$
- Persamaan tersebut berlaku untuk semua pernyataan *adjacent* dari suatu deret, hal tersebut berulang untuk semua nilai k.
- Persamaan seperti diatas disebut linier, orde pertama, koefisien konstan hubungan *recurrence*.
- Orde dari persamaan ditunjukkan oleh pergeseran maksimum antara pernyataan terkait, dalam hal ini berarti 1.

- Berarti, hubungan *recurrence* $X_{k+2} - x_{k+1} - x_k = 1$ mempunyai orde 2
- Sebab pergeseran maksimum antara pernyataan didalam hubungan adalah 2, yaitu dari k ke k+2.

# Pernyataan Awal

- Suatu hubungan *recurrence* dapat digunakan untuk menentukan pernyataan dari suatu deret yang diberikan nilai awal – sama dengan jumlah orde dari persamaan.
- Sebagai contoh, diberikan deret $\{x_k\}$ dimana $x_{k+1} = 3x_k$ dengan nilai awal $x_0 = 2$, tentukanlah deret dari pernyataan tersebut.

Since $x_{k+1} = 3x_k$ where $x_0 = 2$ then

$$x_1 = 3x_0 = 3 \times 2 = 6$$

$$x_2 = 3x_1 = 3 \times 6 = 18$$

$$x_3 = 3x_2 = 3 \times 18 = 54$$

$$\boxed{\{x_k\} = \{2,\ 6,\ 18,\ 54,\ \ldots\}}$$

Similarly, if another sequence has terms that satisfy the second-order recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = 1$$

then the first five terms of the sequence are

$$\{x_k\} = \{0,\ 1,\ \ldots,\ \ldots,\ \ldots,\ \ldots\}$$

Since $x_{k+2} - 3x_{k+1} + 2x_k = 1$ where $x_0 = 0$ and $x_1 = 1$ then

$$x_2 - 3x_1 + 2x_0 = 1 \text{ that is } x_2 - 3 \times 1 + 2 \times 0 = 1 \quad \text{and so } x_2 = 4$$
$$x_3 - 3x_2 + 2x_1 = 1 \text{ that is } x_3 - 3 \times 4 + 2 \times 1 = 1 \quad \text{and so } x_3 = 11$$
$$x_4 - 3x_3 + 2x_2 = 1 \text{ that is } x_4 - 3 \times 11 + 2 \times 4 = 1 \text{ and so } x_4 = 26$$

$$\boxed{\{x_k\} = \{0,\ 1,\ 4,\ 11,\ 26,\ \ldots\}}$$

Try another yourself.

The sequence $\{x_k\}$ has terms that satisfy the second-order recurrence relation

$$x_{k+2} - x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = -1$$

The first six terms of this sequence are

Because

Since $x_{k+2} - x_k = 1$ where $x_0 = 0$ and $x_1 = -1$ then

$x_2 - x_0 = 1$ that is $x_2 - 0 = 1$ and so $x_2 = 1$

$x_3 - x_1 = 1$ that is $x_3 + 1 = 1$ and so $x_3 = 0$

$x_4 - x_2 = 1$ that is $x_4 - 1 = 1$ and so $x_4 = 2$

$x_5 - x_3 = 1$ that is $x_5 - 0 = 1$ and so $x_5 = 1$

Therefore $\{x_k\} = \{0,\ -1,\ 1,\ 0,\ 2,\ 1,\ \ldots\}$

# Solving the recurrence relation

If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the $Z$ transform where $Z\{x_k\} = F(z)$. This is referred to as *solving the recurrence relation*. For example, solve the recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = 1$$

Because this recurrence relation is true for all values of $k$ it can itself be used to form a sequence $\{y_k\}$, namely

$$\{y_k\} = \{x_{k+2} - 3x_{k+1} + 2x_k\} = \{1\}$$

Now, taking the $Z$ transform of both sides of this equation gives

$$Z\{y_k\} = Z\{x_{k+2} - 3x_{k+1} + 2x_k\} = Z\{1\} \text{ that is}$$
$$Z\{x_{k+2}\} - 3Z\{x_{k+1}\} + 2\{x_k\} = Z\{1\}$$

Using the first shift theorem and $Z\{x_k\} = F(z)$ this then becomes

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - 3(zF(z) - zx_0) + 2F(z) = \frac{z}{z-1}$$

Collecting like terms and substituting for the initial terms $x_0 = 0$ and $x_1 = 1$ gives

$$(z^2 - 3z + 2)F(z) - z = \frac{z}{z-1} \text{ so } (z^2 - 3z + 2)F(z) = z + \frac{z}{z-1} = \frac{z^2}{z-1}$$

That is $F(z) = \dfrac{z^2}{(z-1)(z^2 - 3z + 2)} = \dfrac{z^2}{(z-1)^2(z-2)}$

and so $\dfrac{F(z)}{z} = \dfrac{z}{(z-1)^2(z-2)}$

This has the partial fraction breakdown

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}$$

Taking the inverse $Z$ transform of $F(z)$ yields the sequence

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

$$F(z) = -\frac{z}{(z-1)^2} - \frac{2z}{z-1} + \frac{2z}{z-2}$$

Therefore

$$
\begin{aligned}
Z^{-1}F(z) &= -Z^{-1}\left(\frac{z}{(z-1)^2}\right) - 2Z^{-1}\left(\frac{z}{z-1}\right) + 2Z^{-1}\left(\frac{z}{z-2}\right) \\
&= \{-k - 2x_k + 2(2^k)\} \\
&= \{-k - 2 + 2^{k+1}\} \text{ since } x_k = 1
\end{aligned}
$$

Indeed, $\{x_k\} = \{-k - 2 + 2^{k+1}\}$ is the solution to the recurrence relation as can be seen by substituting back

$$
\begin{aligned}
& x_{k+2} - 3x_{k+1} + 2x_k \\
&= \left(-[k+2] - 2 + 2^{[k+2]+1}\right) - 3\left(-[k+1] - 2 + 2^{[k+1]+1}\right) \\
&\quad + 2(-k - 2 + 2^{k+1}) \\
&= (-k - 4 + 8 \times 2^k) - 3(-k - 3 + 4 \times 2^k) + 2(-k - 2 + 2 \times 2^k) \\
&= -k - 4 + 8 \times 2^k + 3k + 9 - 12 \times 2^k - 2k - 4 + 4 \times 2^k \\
&= 1
\end{aligned}
$$

Try one yourself.

The solution of the second-order recurrence relation

$$x_{k+2} - x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = -1 \text{ is } x_k = \ldots\ldots\ldots\ldots$$

Taking the $Z$ transform of the recurrence relation gives

$Z\{x_{k+2} - x_k\} = Z\{1\}$. That is, $Z\{x_{k+2}\} - Z\{x_k\} = Z\{1\}$ so that

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - F(z) = \frac{z}{z-1}.$$

Substituting for $x_0 = 0$ and $x_1 = -1$ gives

$\left(z^2F(z) - z^2x_0 - zx_1\right) - F(z) = \dfrac{z}{z-1}$ where $x_0 = 0$ and $x_1 = -1$ giving

$(z^2 - 1)F(z) + z = \dfrac{z}{z-1}$ so

$F(z) = \dfrac{z}{(z^2-1)(z-1)} - \dfrac{z}{(z^2-1)}$ so

$$\frac{F(z)}{z} = \frac{1}{(z+1)(z-1)^2} - \frac{1}{(z+1)(z-1)}$$
$$= \frac{1-(z-1)}{(z+1)(z-1)^2}$$
$$= \frac{-z+2}{(z+1)(z-1)^2}$$

Separating into partial fractions gives

$$F(z) = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

and

$$Z^{-1}\left\{\frac{z}{z+1}\right\} = \left\{(-1)^k\right\} \text{ so } Z^{-1}\left\{(3/4)\frac{z}{z+1}\right\} = (3/4)\left\{(-1)^k\right\}$$
$$Z^{-1}\left\{\frac{z}{z-1}\right\} = \{1^k\} \text{ so } Z^{-1}\left\{(-3/4)\frac{z}{z-1}\right\} = (-3/4)\{1^k\}$$

$$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = \{k\} \text{ so } Z^{-1}\left\{(1/2)\frac{z}{(z-1)^2}\right\} = (1/2)\{k\}$$

Therefore $\{x_k\} = \left\{(3/4)(-1)^k - (3/4) + (k/2)\right\}$

so that $x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}$

# LATIHAN

Solve the recurrence relation

$$x_{k+2} - 4x_{k+1} + 4x_k = 3 \text{ where } x_0 = 1 \text{ and } x_1 = 0.$$

# Sampling

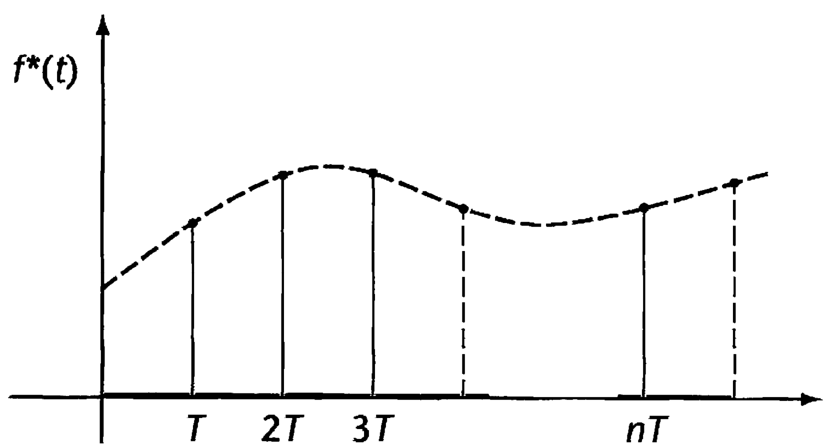
If a continuous function $f(t)$ of time $t$ progresses from $t = 0$ onwards and is measured at every time interval $T$ then what will result is the sequence of values

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\}$$

A new, piecewise continuous function $f^*(t)$ can then be created from the sequence of sampled values such that

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = kT \\ 0 & \text{otherwise} \end{cases}$$

The graph of this new function consists of a series of spikes at the regular intervals $t = kT$



This function can alternatively be described in terms of the delta function $\delta(t)$ as

$$\begin{aligned} f^*(t) &= f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + f(3T)\delta(t-3T) + \ldots \\ &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT) \end{aligned}$$

The Laplace transform of $f^*(t)$ is then given as

$$\begin{aligned} F^*(s) &= L\{f^*(t)\} \\ &= \int_0^{\infty} \{f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \ldots\} e^{-st}\,dt \\ &= f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \ldots \\ &= \sum_{k=0}^{\infty} f(kT)e^{-ksT} \end{aligned}$$

Define a new variable $z = e^{sT}$ and we see that

$$L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$$

which is the $Z$ transform of the sequence$\{f(kT)\}$.

**Example 1**

The function $f(t) = e^{-at}$ is sampled every interval of $T$.

Tentukan transformasi Z dari fungsi hasil pencuplikan tersebut.

Defining $f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t - kT) = \sum_{k=0}^{\infty} e^{-akT}\delta(t - kT)$ then the Laplace transform of $f^*(t)$ is given as

$$F^*(s) = \sum_{k=0}^{\infty} e^{-kaT} e^{-ksT}$$

This means that the $Z$ transform of $\{f(kT)\}$ is

$$F(z) = \sum_{k=0}^{\infty} \frac{e^{-kaT}}{z^k} = \frac{1}{1 - \dfrac{e^{-aT}}{z}} = \frac{z}{z - e^{-aT}}$$

Notice that this agrees with the $Z$ transform of the sequence $\{b^k\}$ $\left(\text{which is } \dfrac{z}{z - b}\right)$ when $b$ is replaced by $e^{-aT}$.

**Example 2**

The function $f(t) = t$ is sampled every interval of $T$.

The $Z$ transform of $\{f(kT)\}$ is $F(z) = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$. Here $f(kT) = kT$ and so

$$
\begin{aligned}
F(z) &= \sum_{k=0}^{\infty} \frac{kT}{z^k} \\
&= T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\right) \\
&= \frac{T}{z}(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \ldots) \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots\} \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{Tz}{(z-1)^2}
\end{aligned}
$$

**Example 3**

The function $f(t) = \cos t$ is sampled every interval of $T$.

$f(t) = \cos t = \dfrac{e^{jT} + e^{-jT}}{2}$ and the $Z$ transform of $\{e^{-kaT}\}$ is

$F(z) = \dfrac{z}{z - e^{-aT}}$.

Therefore the $Z$ transform of $\dfrac{e^{jT} + e^{-jT}}{2}$ is

$$\frac{1}{2}\left(\frac{z}{z - e^{-jT}} + \frac{z}{z - e^{jT}}\right) = \frac{1}{2}\left(\frac{z\left(z - e^{jT}\right) + z\left(z - e^{-jT}\right)}{(z - e^{-jT})(z - e^{jT})}\right)$$

$$= \frac{1}{2}\left(\frac{2z^2 - z\left(e^{jT} + e^{-jT}\right)}{z^2 - [e^{jT} + e^{-jT}]z + 1}\right)$$

$$= \frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}$$

# LATIHAN

The function $f(t) = \sin t$ is sampled at equal intervals of $t = T$. Find the $Z$ transform of the resulting sequence of values.