TRANSFORMASI Z

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Pendahuluan

- Transformasi Laplace digunakan untuk fungsi kontinyu dan bisa digunakan untuk menyelesaikan banyak persamaan diferensial.
- Jika kita dihadapkan dengan fungsi diskrit, maka kita harus menggunakan transformasi Z

Deret

- Deret $\ldots, 3^{-2}, 3^{-1}, 3^0, 3, 3^2, 3^3, \ldots$ adalah pernyataan umum dari bentuk 3^k dan notasi ringkas dari deret tsb bisa digunakan $\{3^k\}_{-\infty}^{\infty}$
- Jumlah

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \ldots + \left(\frac{3}{z}\right)^{-1} + \left(\frac{3}{z}\right)^0 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \ldots$$

Disebut sebagai transformasi Z dari deret, $Z\{3^k\}_{-\infty}^{\infty}$

Dan dinotasikan dengan F(z), kita katakan bahwa

$\{3^k\}_{-\infty}^{\infty}$ and $Z\{3^k\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k$ form a *Z transform pair*.

- untuk keperluan kita, kita hanya akan mempertimbangkan deret kausal dengan bentuk

  $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$

- Untuk menyingkatnya kita bisa menuliskan dengan $\{x_k\}$ dengan menghubungkan pada transformasi Z

$$Z\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}.$$

- Catat bahwa ini adalah definisi dari transformasi Z dari deret $\{x_k\}$
- Sebagai contoh, deret unit impuls $\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$ mempunyai transformasi Z

$$Z\{\delta_k\} = \ldots\ldots\ldots\ldots$$

$$\boxed{Z\{\delta_k\} = 1 \text{ valid for all values of } z}$$

Because

$$Z\{\delta_k\} = \sum_{k=0}^{\infty} \frac{\delta_k}{z_k}$$

$$= 1 + \frac{0}{z} + \frac{0}{z^2} + \ldots = 1$$

Try another.

The sequence $\{u_k\} = \{1, 1, 1, \ldots\} = \{1\}$ is called the *unit step* sequence and has the Z transform

.... provided $|z|$ ....

Next frame

Because

$$\begin{aligned} Z\{u_k\} &= F(z) \\ &= \sum_{k=0}^{\infty} \frac{u_k}{z^k} = \sum_{k=0}^{\infty} \frac{1}{z^k} \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \end{aligned}$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= \frac{1}{1 - \frac{1}{z}} \text{ provided } \left|\frac{1}{z}\right| < 1 \\ &= \frac{z}{z-1} \text{ provided } |z| > 1 \end{aligned}$$

And another.

Given the causal sequence $\{x_k\} = \{1, a, a^2, a^3, a^4, \dots\} = \{a^k\}$

the Z transform is …………

$$Z\{a^k\} = \sum_{k=0}^{\infty} \frac{a^k}{z^k}$$
$$= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$
$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ which is valid for $|x| < 1$ then

$$F(z) = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots$$
$$= \frac{1}{1 - \frac{a}{z}} \text{ provided } \left|\frac{a}{z}\right| < 1.$$

That is, multiplying numerator and denominator by z

$$F(z) = \frac{z}{z-a} \text{ provided } |z| > |a|$$

Let's try another. The sequence $\{x_k\} = \{0, 1, 2, 3, 4, \ldots\} = \{k\}$ has the Z transform

$$Z\{k\} = F(z) = \ldots\ldots\ldots\ldots$$

$$\begin{aligned} Z\{k\} &= F(z) \\ &= \sum_{k=0}^{\infty} \frac{x_k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\ &= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ldots \end{aligned}$$

By comparing this sequence with the derivative of $(1 - x)^{-1}$ and its series representation, this sequence can be written as a rational expression in z as $F(z) = \ldots\ldots\ldots\ldots$

$$F(z) = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ldots$$

Comparing this with the series expansion

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + \cdots) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(1 - x)^{-1} = \frac{1}{(1 - x)^2} \end{aligned}$$

then we can see that by multiplying $F(z)$ by z

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots = \frac{1}{(1 - 1/z)^2}$$

so, dividing both sides by z gives

$$F(z) = \frac{1}{z(1 - 1/z)^2} = \frac{z}{(z - 1)^2}$$

Tabel Transformasi Z

Sequence	Transform $F(z)$	Permitted values of z
$\{\delta_k\} = \{1, 0, 0, \ldots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, \ldots\}$	$\frac{z}{z-1}$	$\|z\| > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\frac{z}{(z-a)}$	$\|z\| > \|a\|$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$

Sifat-sifat Transformasi Z

1. Linieritas

   Transformasi Z adalah transformasi linier, yaitu jika a dan b adalah konstanta maka;

   $$Z(a\{x_k\} + b\{y_k\}) = aZ\{x_k\} + bZ\{y_k\}$$

   Sebagai contoh;

   $Z\{k\} = \dfrac{z}{(z-1)^2}$ from the table and, also from the table,

   $Z\{a^k\} = \dfrac{z}{z-a}$ so when $a = e^{-2}$,

   $$Z\left\{e^{-2k}\right\} = \frac{z}{z - e^{-2}}$$

   Consequently, the Z transform of $3\{k\} - 5\left\{e^{-2k}\right\}$ is ....

$$Z\left(3\{k\} - 5\left\{e^{-2k}\right\}\right) = 3Z\{k\} - 5Z\left\{e^{-2k}\right\}$$

$$= \frac{3z}{(z-1)^2} - \frac{5z}{(z-e^{-2})}$$

$$= \frac{3z\left(z-e^{-2}\right) - 5z(z-1)^2}{(z-1)^2(z-e^{-2})}$$

$$\frac{3z^2 - 3ze^{-2} - 5z^3 + 10z^2 - 5z}{(z-1)^2(z-e^{-2})}$$

$$= \frac{-5z^3 + 13z^2 - z\left(3e^{-2} + 5\right)}{(z-1)^2(z-e^{-2})}$$

2 ***First shift theorem*** **(shifting to the left)**

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

is the Z transform of the sequence that has been shifted by m places to the left. For example

$$Z\{x_{k+1}\} = zF(z) - zx_0$$

$$Z\{x_{k+2}\} = z^2 F(z) - z^2 x_0 - zx_1$$

These will be used later when solving difference equations. Note the similarity between these results and the Laplace transforms for the first and second derivatives for continuous functions.

For example, given that $Z\{4^k\} = \dfrac{z}{z-4}$ then

$$Z\{4^{k+3}\} = \ldots\ldots\ldots\ldots$$

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

so

$$Z\{4^{k+3}\} = z^3 Z\{4^k\} - \left[z^3 4^0 + z^2 4^1 + z4^2\right] \text{ where } Z\{4^k\} = \frac{z}{z-4}$$

$$= z^3 \frac{z}{z-4} - \left[z^3 + 4z^2 + 16z\right]$$

$$= \frac{z^4}{z-4} - \left[z^3 + 4z^2 + 16z\right]$$

$$= \frac{z^4 - (z^3 + 4z^2 + 16z)(z-4)}{z-4}$$

$$= \frac{z^4 - (z^4 - 64z)}{z-4}$$

$$= \frac{64z}{z-4}$$

In this way we have derived the Z transform of the sequence $\{64, 256, 1024, \ldots\}$ by shifting the sequence $\{1, 4, 16, 64, 256, \ldots\}$ three places to the left and losing the first three terms.

Try another. Given that $Z\{k\} = \frac{z}{(z-1)^2}$ then

$$Z\{(k+1)\} = \ldots\ldots\ldots\ldots$$

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

so

$$Z\{k+1\} = z\frac{z}{(z-1)^2} - [z \times 0]$$

$$= \frac{z^2}{(z-1)^2}$$

3 ***Second shift theorem*** (shifting to the right)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

For example, given that $Z\{x_k\} = \dfrac{z}{z-1}$ then

$$Z\{x_{k-3}\} = \ldots\ldots\ldots\ldots$$

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

so

$$Z\{x_{k-3}\} = z^{-3}\frac{z}{z-1} = \frac{1}{z^2(z-1)}$$

In this way we have derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, \ldots\}$ by shifting the sequence $\{1, 1, 1, 1, \ldots\}$ three places to the right and defining the first three terms as zeros.

Try this one. The sequence $\{x_k\}$ with Z transform

$$Z\{x_k\} = \frac{1}{(z-a)}, \text{ where } a \text{ is a constant, is } \{\ldots\ldots\ldots\ldots\}$$

From the table of transforms the nearest transform to the one in question is $\dfrac{z}{(z-a)}$ which is the Z transform of $\{a^k\}$. Now

$$\begin{aligned}\frac{1}{(z-a)} &= \frac{1}{z} \times \frac{z}{(z-a)} \\ &= z^{-1}F(z) \quad \text{where } F(z) = Z\{a^k\}\end{aligned}$$

and so

$$\frac{1}{(z-a)} = Z\{a^{k-1}\}$$

which is the Z transform of $\{a^k\}$, shifted one place to the right.

4 *Translation*

If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

For example, $Z\{k\} = \dfrac{z}{(z-1)^2}$ so that $Z\{2^k k\} = \ldots\ldots\ldots\ldots$

Since $Z\{k\} = \dfrac{z}{(z-1)^2} = F(z)$ then by the translation property

$$\begin{aligned} Z\{2^k k\} &= F(2^{-1}z) \\ &= \frac{2^{-1}z}{(2^{-1}z-1)^2} \\ &= \frac{2z}{(z-2)^2} \end{aligned}$$

5 *Final value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\lim_{k\to\infty} x_k = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \lim_{k\to\infty} x_k \text{ exists.}$$

For example, the sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}.$$

Now

$$\lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \lim_{z\to 1}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

and

$$\lim_{k\to\infty}\left\{\left(\frac{1}{2}\right)^k\right\} = 0 \text{ which confirms the final value theorem.}$$

Using the final value theorem the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2 + 2z}{(z-1)(5z-1)^2} \text{ is } \ldots\ldots\ldots\ldots$$

$$\begin{aligned}
\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} &= \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)\frac{10z^2 + 2z}{(z-1)(5z-1)^2}\right\} \\
&= \underset{z\to 1}{Lim}\left\{\frac{10z+2}{(5z-1)^2}\right\} \\
&= \frac{12}{16} \\
&= 0{\cdot}75
\end{aligned}$$

6 The initial value theorem

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\{F(z)\}$$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and $\underset{z\to\infty}{Lim}\,F(z) = \underset{z\to\infty}{Lim}\dfrac{z}{z-a} = \underset{z\to\infty}{Lim}\dfrac{1}{1} = 1$ by L'Hôpital's rule. Furthermore $x_0 = a^0 = 1$ so demonstrating the validity of the theorem.

7 The derivative of the transform

If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$

This is easily proved.

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k} \text{ and so } F'(z) = \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k}$$

$$= -\frac{1}{z}Z\{kx_k\}$$

and so $-zF'(z) = Z\{kx_k\}$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and so the sequence $\{ka^k\}$ has Z transform

$$Z\{kx_k\} = -zF'(z) = \ldots\ldots\ldots\ldots$$

$$-zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Test exercise 5

1 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-1)^k$.

2 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = 4k - 2a^k$.

3 Find the Z transform of the causal sequences:

(a) $\{k-3\}$

(b) $\{5^{k+2}\}$

Inverse transforms

If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}$$

There are many times when, given the Z transform of a sequence, it is not possible to immediately read off the sequence from the Table of transforms. Instead some manipulation may be required and, as with Laplace transforms, very often this involves using partial fractions.

Example

The sequence $\{x_k\}$ has Z transform $F(z) = \dfrac{z}{z^2 - 5z + 6}$. To find the inverse transform, and hence the sequence, we recognise that the denominator can be factorised and separated into partial fractions as

$$\begin{aligned} F(z) &= \frac{z}{z^2 - 5z + 6} \\ &= \frac{z}{(z-2)(z-3)} \\ &= \frac{A}{z-2} + \frac{B}{z-3} \\ &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)} \end{aligned}$$

Equating numerators gives $z = A(z-3) + B(z-2)$, giving $A + B = 1$ and $-3A - 2B = 0$. From these two equations we find that $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $Z\{a^k\} = \dfrac{z}{z-a}$. Therefore if we write

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

$$F(z) = \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2}$$

$$= 3 \times z^{-1} Z\{3^k\} - 2 \times z^{-1} Z\{2^k\}$$

and so

$$Z^{-1}F(z) = 3 \times \{3^{k-1}\} - 2 \times \{2^{k-1}\} \text{ by the second shift theorem}$$

$$= \{3^k\} - \{2^k\}$$

$$= \{3^k - 2^k\} \text{ giving } x_k = 3^k - 2^k$$

There is a simpler way of doing this without employing the second shift theorem. Recognising that z appears in the numerator of $F(z)$, we consider instead the partial fraction breakdown of $\frac{F(z)}{z}$

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z} \times \frac{z}{z^2 - 5z + 6} \\ &= \frac{1}{z^2 - 5z + 6} \\ &= \frac{1}{(z-2)(z-3)} \\ &= \frac{A}{z-2} + \frac{B}{z-3} \\ &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators gives $1 = A(z-3) + B(z-2)$, giving

[z]: $A + B = 0$

[CT]: $-3A - 2B = 1$ with solution $A = -1$ and $B = 1$. So that

$$\frac{F(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2} \text{ that is}$$

$$F(z) = \frac{z}{z-3} - \frac{z}{z-2}$$

$$= Z\{3^k\} - Z\{2^k\} \text{ and so}$$

$$Z^{-1}F(z) = \{3^k\} - \{2^k\}$$

$$= \{3^k - 2^k\}$$

Thus the use of the second shift theorem is avoided.

So try one yourself. The sequence $\{x_k\}$ has Z transform

$$F(z) = \frac{5z}{(z^2 - 4z + 4)(z + 2)}$$

therefore $\{x_k\} = \ldots\ldots\ldots\ldots$

$$\frac{F(z)}{z} = \frac{1}{z} \times \frac{5z}{(z^2 - 4z + 4)(z + 2)}$$

$$= \frac{5}{(z-2)^2(z+2)}$$

$$= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z+2}$$

$$= \frac{A(z+2) + B(z-2)(z+2) + C(z-2)^2}{(z-2)^2(z+2)}$$

Equating numerators gives $5 = A(z+2) + B(z^2 - 4) + C(z^2 - 4z + 4)$

$$[z^2]: \quad B + C = 0$$
$$[z]: \quad A - 4C = 0$$
$$[\text{CT}]: \quad 2A - 4B + 4C = 5$$

with solution $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2} \text{ giving}$$

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2} \text{ and so}$$

$$Z^{-1}F(z) = \frac{5}{8} \times \{k2^k\} - \frac{5}{16} \times \{2^k\} + \frac{5}{16} \times \left\{(-2)^k\right\}$$
$$= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\}$$

LATIHAN

Find the inverse Z transformation of

$$F(z) = \frac{z^2(z-3)}{(z^2-2z+1)(z-2)}.$$