OSILATOR HARMONIK

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Harmonic oscillators

- Jika posisi sebuah sistem pada waktu t dideskripsikan dengan f(t) dimana f(t) memenuhi persamaan diferensial

  $af''(t) + bf(t) = 0$, $f(0) = \alpha$ and $f'(0) = \beta$

  (and where a and b have the same sign)

- Maka, dengan menggunakan transformasi laplace diperoleh

  $$L\{af''(t) + bf(t)\} = L\{0\}$$

  That is

  $$a\left[s^2F(s) - s\alpha - \beta\right] + b\left[F(s)\right] = 0$$

  Collecting like terms gives

  $$\left(as^2 + b\right)F(s) = s\alpha + \beta$$

- Diperoleh

$$F(s) = \frac{s\alpha + \beta}{as^2 + b}$$

Therefore $F(s) = \frac{s(\alpha/a)}{s^2 + (b/a)} + \frac{\beta/a}{s^2 + (b/a)}$ and so

$$f(t) = \frac{\alpha}{a}\cos\sqrt{\frac{b}{a}}t + \frac{\beta}{a}\sin\sqrt{\frac{b}{a}}t$$

- Sistem menghasilkan harmonik sederhana, gerak osilasi dengan frekuensi;

$\sqrt{\frac{b}{a}}$ radians per unit of time and with period $\frac{2\pi}{\sqrt{b/a}} = 2\pi\sqrt{\frac{a}{b}}$. It is called an **harmonic oscillator**. Let's try some examples.

Example 1

Find the solution to the harmonic oscillator

$f''(t) + 16f(t) = 0$ where $f(0) = 1$ and $f'(0) = 0$

Taking Laplace transforms $L\{f''(t) + 16f(t)\} = L\{0\}$.

That is $s^2F(s) - s + 16F(s) = 0$ and so

$$F(s) = \frac{s}{s^2 + 16}$$

This means that

$$\boxed{f(t) = \cos 4t}$$

Because

$F(s) = \dfrac{s}{s^2 + 16} = \dfrac{s}{s^2 + 4^2}$ so $f(t) = \cos 4t$ from the Table of Laplace transforms on page 68.

The motion of this system is then periodic with frequency 4 radians per unit of time and with period $2\pi/4 = \pi/2$ units of time.

Example 2

The frequency and period of the harmonic oscillator whose position $f(t)$ satisfies the differential equation

$$5f''(t) + 10f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 4$$

Taking Laplace transforms gives

$$L\{5f''(t) + 10f(t)\} = L\{0\} \text{ that is } 5s^2F(s) - 4 + 10F(s) = 0 \text{ so that}$$

$$F(s) = \frac{4}{5s^2 + 10} = \frac{4/5}{s^2 + 2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{2\sqrt{2}}{5} \sin \sqrt{2}t$$

This is periodic with frequency $\sqrt{2}$ radians per unit of time and period $2\pi/\sqrt{2} = \sqrt{2}\pi$ units of time.

Notice that the amplitude of the motion is $\frac{2\sqrt{2}}{5}$.

Damped motion

Consider the equation

$$5f''(t) + 5f'(t) + 10f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 4$$

Solving the differential equation gives

$$f(t) = \ldots\ldots\ldots\ldots$$

Taking Laplace transforms gives

$$L\{5f''(t) + 5f'(t) + 10f(t)\} = L\{0\} \text{ that is}$$

$$5\left(s^2F(s) - 4\right) + 5sF(s) + 10F(s) = 0$$

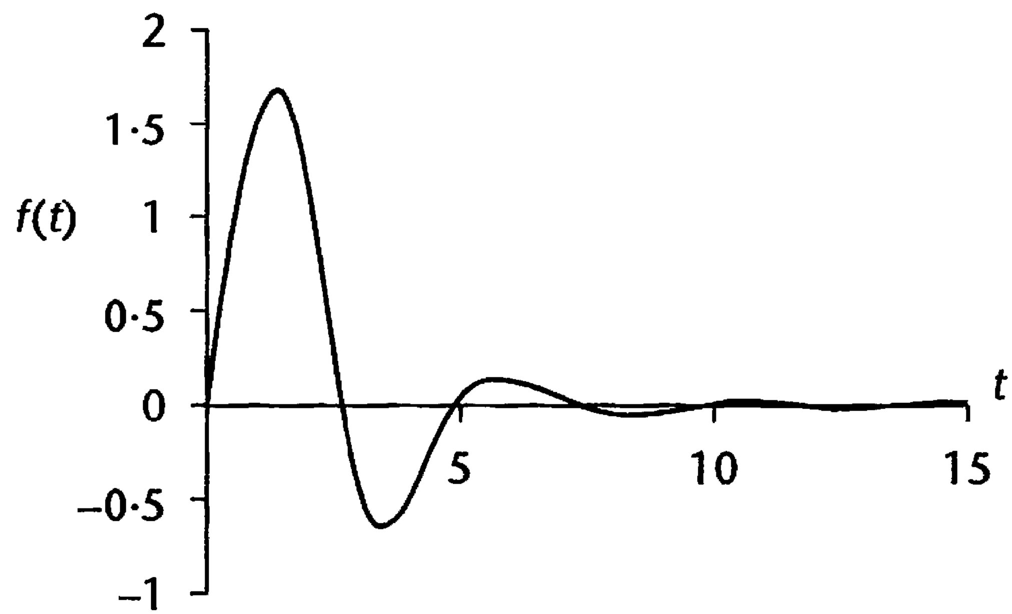
so that

$$F(s) = \frac{20}{5s^2 + 5s + 10} = \frac{4}{s^2 + s + 2} = \frac{4}{(s + 1/2)^2 + (\sqrt{7}/2)^2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{8}{\sqrt{7}} e^{-t/2} \sin\left(\sqrt{7}t/2\right)$$

This is periodic with frequency 1 radian per unit of time and period 2π units of time but with an amplitude that is decreasing with time. The graph of this function is as follows



The effect of the $5f'(t)$ in the differential equation is to introduce **damping** into the oscillatory motion so causing the oscillations to decay. Let's try another example.

Example 3

Consider the equation

$$5f''(t) + f'(t) + 10f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 4$$

This equation is again similar to the previous equation but with a smaller damping term of $f'(t)$ instead of $5f'(t)$. Then here
Taking Laplace transforms gives

$$L\{5f''(t) + f'(t) + 10f(t)\} = L\{0\} \text{ that is}$$

$$5\left(s^2F(s) - 4\right) + sF(s) + 10F(s) = 0$$

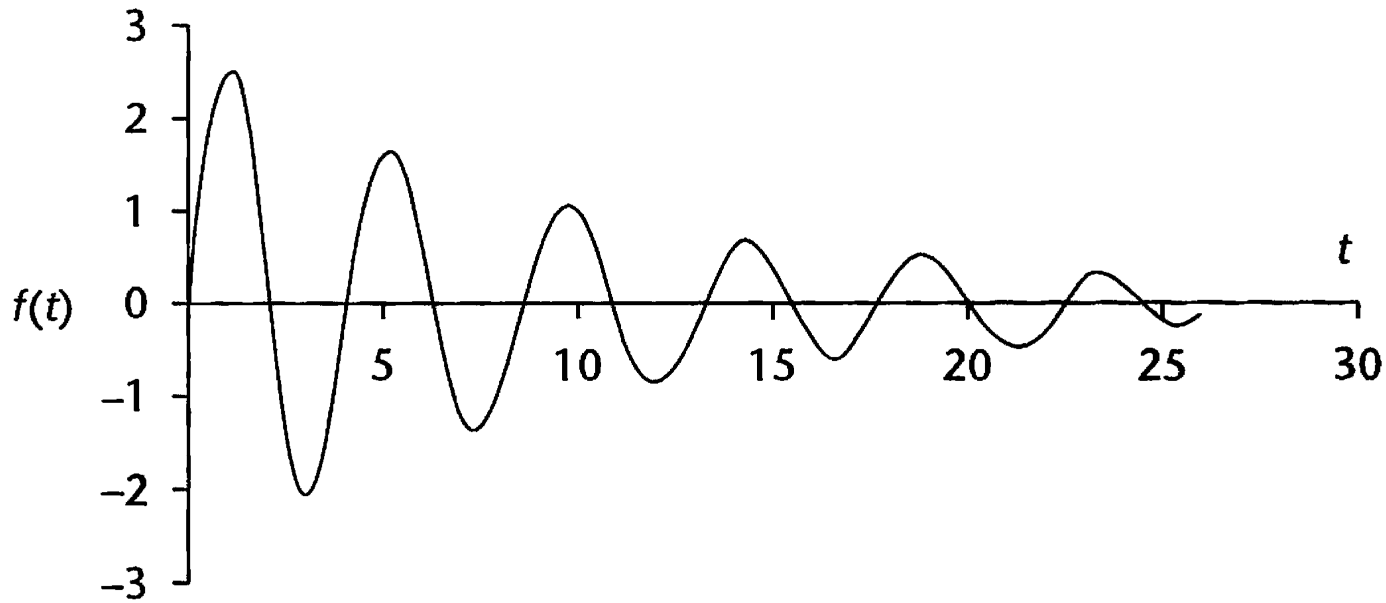
so that

$$F(s) = \frac{20}{5s^2 + 1s + 10} = \frac{4}{s^2 + 0{\cdot}2s + 2} = \frac{4}{(s + 0{\cdot}1)^2 + 1{\cdot}99}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{4}{\sqrt{1{\cdot}99}} e^{-0.1t} \sin \sqrt{1{\cdot}99}t$$

This is periodic with frequency $\sqrt{1{\cdot}99}$ radians per unit of time and period $2\pi/\sqrt{1{\cdot}99}$ units of time and with an amplitude that is decreasing with time. The graph of this function is as follows



Again, the effect of the $f'(t)$ in the differential equation is to introduce damping into the oscillatory motion so causing it to decay. Also because the coefficient of $f'(t)$ is smaller in this example, the damping is less severe.

Latihan

Find the frequency, periodic time and solution for each of the following harmonic oscillators.

(a) $f''(t) + f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 1$

(b) $6f''(t) + 2f'(t) + 9f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 3$.

Forced harmonic motion with damping

The equation

$$f''(t) + f'(t) + f(t) = e^t \text{ where } f(0) = 0 \text{ and } f'(0) = 0$$

we know would represent damped harmonic motion were it not for the exponential on the right-hand side. To see the effect of the exponential we solve the equation.

Taking Laplace transforms we see that

$$L\{f''(t) + f'(t) + f(t)\} = L\{e^t\} \text{ that is } (s^2 + s + 1)F(s) = \frac{1}{s-1} \text{ so}$$

$$F(s) = \frac{1}{(s-1)(s^2+s+1)}$$

Separating into partial fractions gives

$$F(s) = \ldots\ldots\ldots\ldots$$

$$\frac{1}{(s-1)(s^2+s+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+s+1)}$$

$$= \frac{A\left(s^2+s+1\right)+(Bs+C)(s-1)}{(s-1)(s^2+s+1)}$$

Equating numerators and then comparing coefficients of powers of s gives

$$1 = A\left(s^2+s+1\right)+(Bs+C)(s-1)$$

$[s^2]$:	$0 = A + B$	(1)	So (2) + (3):	$1 = 2A - B$
$[s]$:	$0 = A - B + C$	(2)	$2 \times (1)$:	$0 = 2A + 2B$
[CT]:	$1 = A - C$	(3)	Therefore:	$-1 = 3B$

so $B = -1/3 = -A$ and $C = -2/3$

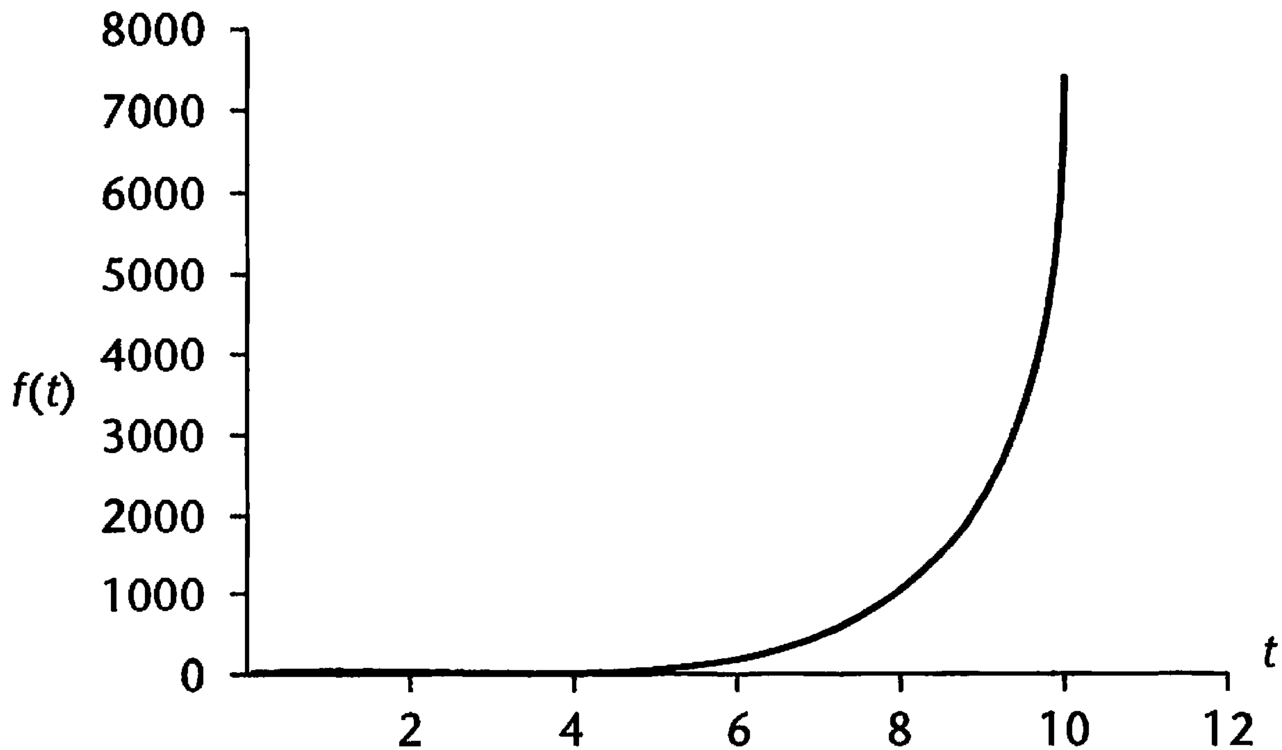
Thus $F(s) = \dfrac{1}{(s-1)(s^2+s+1)} = \dfrac{1}{3(s-1)} - \dfrac{s+2}{3(s^2+s+1)}$

$$= \frac{1}{3(s-1)} - \frac{s+\frac{1}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)} - \frac{\frac{3}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)}$$

So

$$f(t) = \frac{e^t}{3} - \frac{1}{3}e^{-t/2}\left(\cos\tfrac{\sqrt{3}}{2}t + \sqrt{3}\sin\tfrac{\sqrt{3}}{2}t\right)$$

from the Table of Laplace transforms on page 68.



- Perlu dicatat bahwa persamaan $\frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)$ mewakili gerak hamonik teredam dan disebut persamaan transien, sedangkan persamaan $e^t/3$ mewakili persamaan steady-state.

Try another one for yourself. The transient and steady-state terms of the system described by the differential equation

$$f''(t) + 2f'(t) + 5f(t) = e^{2t} \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

Taking Laplace transforms, $L\{f''(t) + 2f'(t) + 5f(t)\} = L\{e^{2t}\}$. That is

$\left[s^2F(s) - 1\right] + 2sF(s) + 5F(s) = \frac{1}{s-2}$, that is

$$(s^2 + 2s + 5)F(s) = 1 + \frac{1}{s-2} = \frac{s-1}{s-2}$$

So that $F(s) = \frac{s-1}{(s-2)(s^2+2s+5)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+5}$. Hence

$s - 1 = A(s^2 + 2s + 5) + (Bs + C)(s - 2)$. Equating powers of s gives

$[s^2]$: $\quad 0 = A + B$

$[s]$: $\quad 1 = 2A - 2B + C$

[CT]: $\quad -1 = 5A - 2C$

Solving these three equations gives $A = 1/13$, $B = -1/13$ and $C = 9/13$ so that

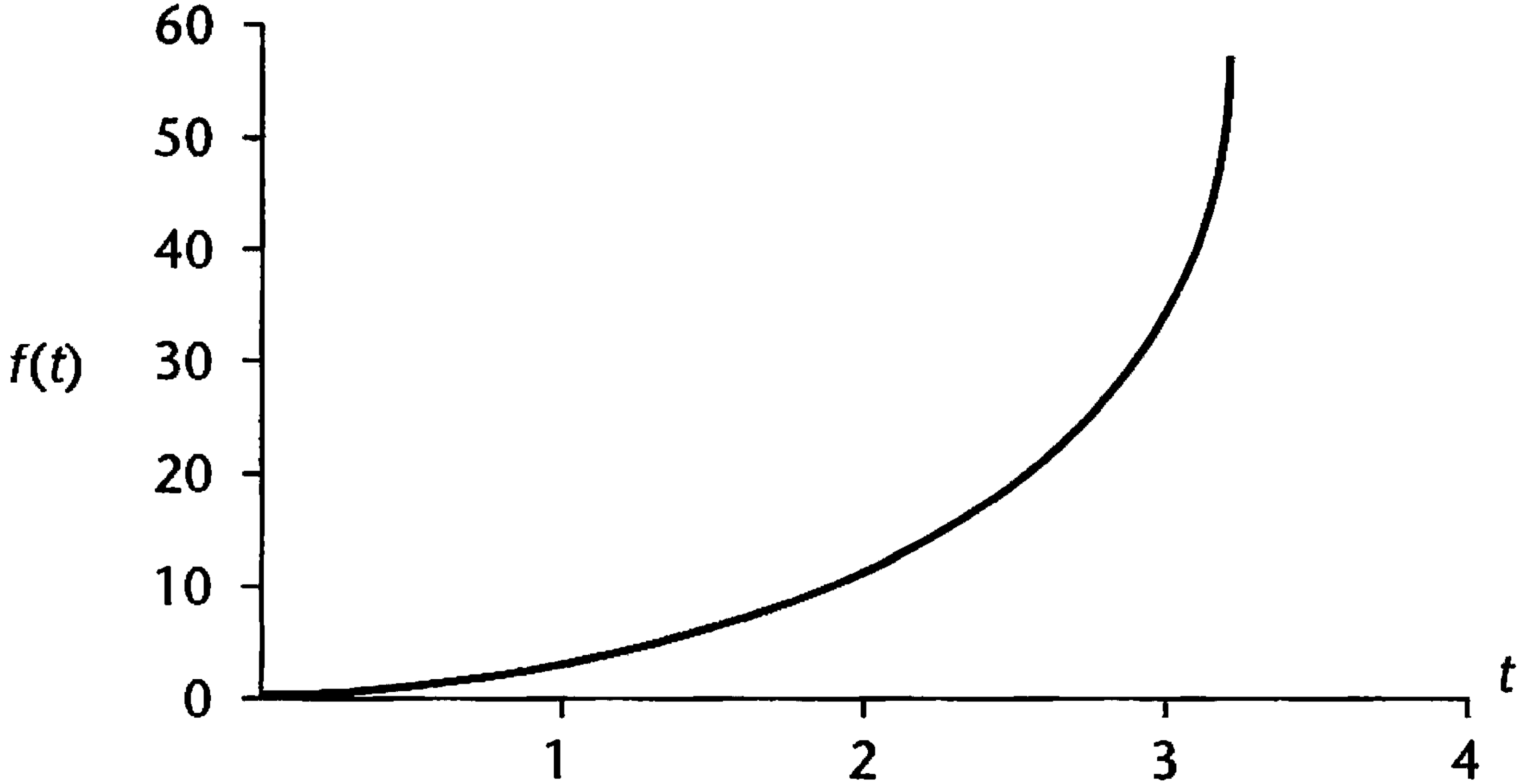
$$F(s) = \frac{1}{13(s-2)} - \frac{s-9}{13(s^2+2s+5)}$$

$$= \frac{1}{13(s-2)} - \frac{s-9}{13\left((s+1)^2+2^2\right)}.$$ That is

$$F(s) = \frac{1}{13(s-2)} - \frac{s+1}{13\left((s+1)^2+2^2\right)} + \frac{10}{13\left((s+1)^2+2^2\right)}$$

Therefore

$$f(t) = \frac{1}{13}e^{2t} - \frac{1}{13}e^{-t}\cos 2t + \frac{5}{13}e^{-t}\sin 2t$$


60
50
40
30
20
10
0
f(t)
1
2
3
4
t

Latihan

Find the transient and steady-state solutions of the forced harmonic oscillator

$f''(t) + 2f'(t) + 3f(t) = 4e^{5t}$ given that $f(0) = -2$ and $f'(0) = 6$.

Resonance

- Persamaan diferensial dengan fungsi pada sisi kanan disebut persamaan diferensial tidak homogen.
- PD tsb mewakili sistem yang mempunyai sifat f(t) yang ditentukan oleh strutur dari sisi kiri dan fungsi ‘forcing’ pada sisi kanan.
- Jika sistem tidak meredam dan tidak menguatkan yang mempunyai sifat periodik diberikan fungsi ‘forcing’ periodik yang mempunyai periode yang sama, maka resonansi akan terjadi dan sistem akan mempunyai sifat periodik dengan amplitudo yang makin membesar.

The differential equation

$f''(t) + f(t) = 0$ where $f(0) = 0$ and $f'(0) = 1$

represents an undamped, unforced system with behaviour

$$\boxed{f(t) = \sin t}$$

Because

Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{0\}$ that is $s^2F(s) - 1 + F(s) = 0$ so that

$F(s) = \dfrac{1}{s^2 + 1}$ giving $f(t) = \sin t$

If the forcing term $-2\sin t$ is applied to the right-hand side of the equation it has the same period as the natural frequency of the system being forced and so resonance will set in. The differential equation to solve is then

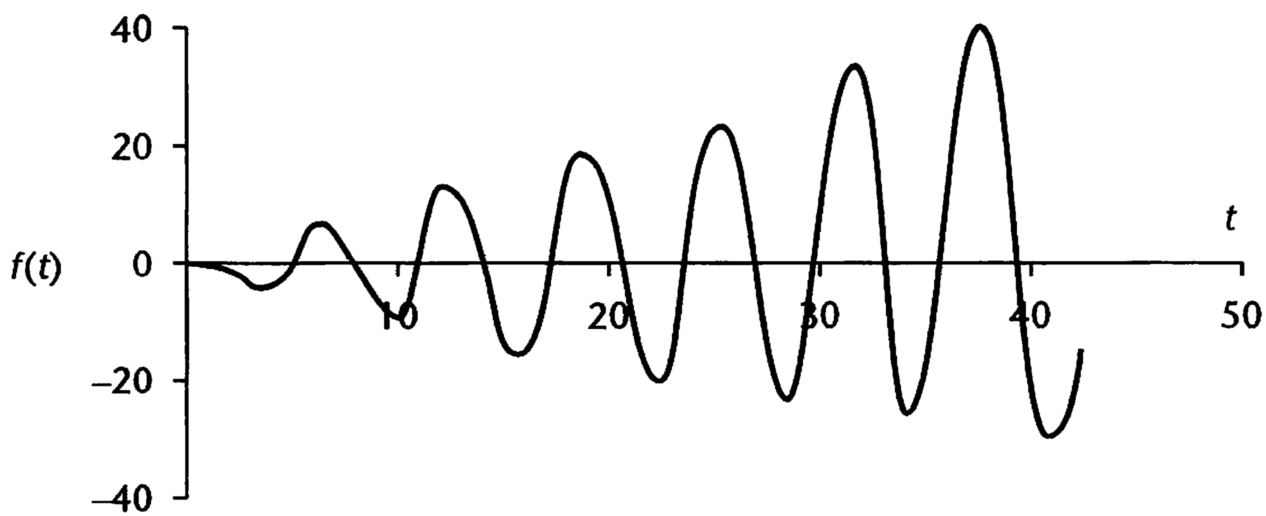
$f''(t) + f(t) = -2\sin t$ where $f(0) = 0$ and $f'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{-2\sin t\}$ that is $s^2F(s) - 1 + F(s) = -\dfrac{2}{s^2+1}$

so that $F(s) = \dfrac{1}{s^2+1} - \dfrac{2}{(s^2+1)^2}$ giving $F(s) = \dfrac{s^2-1}{(s^2+1)^2}$. Now, the Laplace transform of $\cos t$ is $\dfrac{s}{s^2+1}$ and $\left(\dfrac{s}{s^2+1}\right)' = -\dfrac{s^2-1}{(s^2+1)^2}$.

Therefore $\quad f(t) = t\cos t$



The system undergoes periodic behaviour with an increasing amplitude.