TRANSFORMASI LAPLACE DIRAC DELTA-UNIT IMPULS

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The Dirac delta – the unit impulse

- Jika f(t) menyatakan sebuah fungsi, maka dirac delta δ(t) didefinisikan sebagai:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$$

- Defenisi yang paling dekat untuk menyatakan dirac delta adalah:

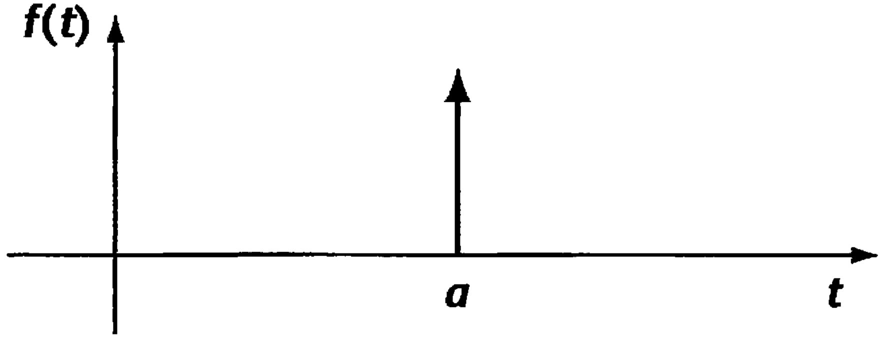
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

- Jika p < a < q, maka;

$$\int_{p}^{q} \delta(t-a)\,\mathrm{d}t = 1$$

Pernyataan Secara Grafik

- Dirac delta atau unit impuls δ(t-a) dinyatakan sebagai sumbu horizontal dengan garis vertikal yang panjangnya tak hingga pada saat t = a



So far, then, we have

(a) $\int_p^q \delta(t-a)\,\mathrm{d}t = 1$

(b) $\int_p^q f(t)\cdot\delta(t-a)\,\mathrm{d}t = f(a)$

provided, in each case, that $p < a < q$.

Example 1

To evaluate $\int_1^3 (t^2 + 4) \cdot \delta(t - 2)\,dt$.

The factor $\delta(t-2)$ shows that the impulse occurs at $t = 2$, i.e. $a = 2$.

$$f(t) = t^2 + 4 \qquad \therefore\ f(a) = f(2) = 4 + 4 = 8$$

$$\therefore\ \int_1^3 (t^2 + 4) \cdot \delta(t - 2)\,dt = f(2) = 8$$

Example 2

To evaluate $\int_0^\pi \cos 6t \cdot \delta(t - \pi/2)\,dt$.

$$\int_0^\pi \cos 6t \cdot \delta(t - \pi/2)\,dt = f(\pi/2) = \cos 3\pi = -1$$

and in the same way

(a) $\int_0^6 5 \cdot \delta(t-3)\,\mathrm{d}t = 5 \times 1 = 5$

(b) $\int_2^5 e^{-2t} \cdot \delta(t-4)\,\mathrm{d}t = \left[e^{-2t}\right]_{t=4} = e^{-8}$

(c) $\int_0^\infty (3t^2 - 4t + 5) \cdot \delta(t-2)\,\mathrm{d}t = 12 - 8 + 5 = 9$

Transformasi Laplace dari δ(t-a)

We have already shown that

$$\int_p^q f(t)\cdot\delta(t-a)\,\mathrm{d}t = f(a) \qquad p<a<q$$

Therefore, if $p=0$ and $q=\infty$

$$\int_0^\infty f(t)\cdot\delta(t-a)\,\mathrm{d}t = f(a)$$

Hence, if $f(t)=e^{-st}$, this becomes

$$\int_0^\infty e^{-st}\cdot\delta(t-a)\,\mathrm{d}t = L\{\delta(t-a)\}$$

$$= e^{-as}$$

for $a=0$, $L\{\delta(t-a)\} = L\{\delta(t)\} = e^0 = 1$

$$\therefore\; L\{\delta(t)\} = 1$$

Finally, let us deal with the more general case of $L\{f(t) \cdot \delta(t-a)\}$.

We have $L\{f(t) \cdot \delta(t-a)\} = \int_0^{\infty} e^{-st} \cdot f(t) \cdot \delta(t-a)\, dt$. Now the integrand $e^{-st} \cdot f(t) \cdot \delta(t-a) = 0$ for all values of t except at $t = a$ at which point $e^{-st} = e^{-as}$, and $f(t) = f(a)$.

$$\therefore \; L\{f(t) \cdot \delta(t-a)\} = f(a) \cdot e^{-as} \int_0^{\infty} \delta(t-a)\, dt$$

$$= f(a) \cdot e^{-as}(1)$$

$$\therefore \; L\{f(t) \cdot \delta(t-a)\} = f(a)e^{-as}$$

Therefore

(a) $L\{6 \cdot \delta(t-4)\}$ $\quad a = 4, \quad \therefore \; L\{6 \cdot \delta(t-4)\} = 6e^{-4s}$

(b) $L\{t^3 \cdot \delta(t-2)\}$ $\quad a = 2, \quad \therefore \; L\{t^3 \cdot \delta(t-2)\} = 8e^{-2s}$

Similarly

(c) $L\{\sin 3t \cdot \delta(t - \pi/2)\} = [\sin 3t]_{t=\pi/2} \cdot e^{-\pi s/2} = -e^{-\pi s/2}$

and

(d) $L\{\cosh 2t \cdot \delta(t)\} = [\cosh 2t]_{t=0} \cdot e^{0} = \cosh 0 \cdot (1) = 1$

So our main conclusions so far are as follows.

(1) $\int_p^q \delta(t-a)\,\mathrm{d}t = 1$ provided $p < a < q$

(2) $\int_p^q f(t) \cdot \delta(t-a)\,\mathrm{d}t = f(a)$ provided $p < a < q$

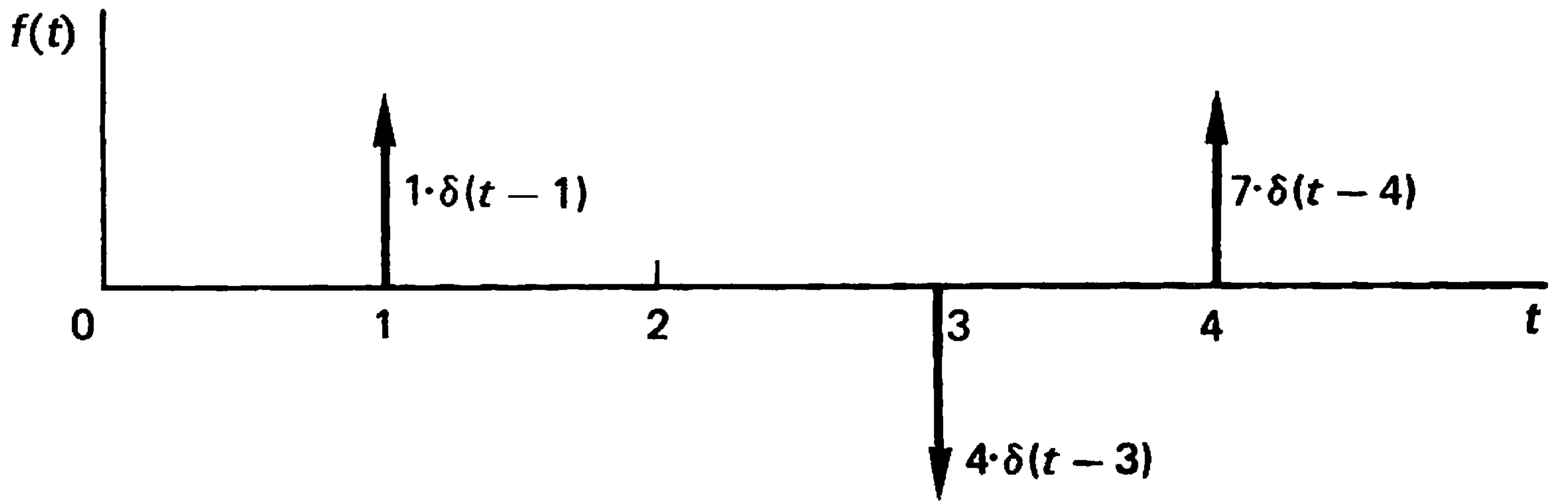
(3) $L\{\delta(t-a)\} = e^{-as}$

(4) $L\{\delta(t)\} = 1$

(5) $L\{f(t) \cdot \delta(t-a)\} = f(a) \cdot e^{-as}$

Example

Impulses of 1, 4, 7 units occur at $t = 1$, $t = 3$ and $t = 4$ respectively, in the directions shown.



Write down an expression for $f(t)$ and determine its Laplace transform.

We have $f(t) = 1 \cdot \delta(t-1) - 4 \cdot \delta(t-3) + 7 \cdot \delta(t-4)$.

Then $L\{f(t)\} = e^{-s} - 4e^{-3s} + 7e^{-4s}$

Differential equations involving the unit impulse

Example 1

A system has the equation of motion

$$\ddot{x} + 6\dot{x} + 8x = g(t)$$

where $g(t)$ is an impulse of 4 units applied at $t = 5$. At $t = 0$, $x = 0$ and $\dot{x} = 3$. Determine an expression for the displacement x in terms of t.

The impulse of 4 units is applied at $t = 5$. $\therefore\ g(t) = 4 \cdot \delta(t-5)$.

$$\therefore\ \ddot{x} + 6\dot{x} + 8x = 4 \cdot \delta(t-5) \quad \text{At } t = 0,\ x = 0,\ \dot{x} = 3.$$

Taking Laplace transforms this differential equation becomes

$$\left(s^2\bar{x} - sx_0 - x_1\right) + 6(s\bar{x} - x_0) + 8\bar{x} = 4e^{-5s}$$

Now $x_0 = 0$; $x_1 = 3$

$$\therefore\ s^2\bar{x} - 3 + 6s\bar{x} + 8\bar{x} = 4e^{-5s}$$

$$\therefore\ (s^2 + 6s + 8)\bar{x} = 3 + 4e^{-5s}$$

$$\therefore\ \bar{x} = (3 + 4e^{-5s})\frac{1}{(s+2)(s+4)}$$

Writing $\dfrac{1}{(s+2)(s+4)}$ in partial fractions, we get

$$\bar{x} = \frac{3}{2}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\} + 2\left\{\frac{e^{-5s}}{s+2} - \frac{e^{-5s}}{s+4}\right\}$$

Taking inverse transforms

$$x = \frac{3}{2}\{e^{-2t} - e^{-4t}\} + 2\left\{e^{-2(t-5)} \cdot u(t-5) - e^{-4(t-5)} \cdot u(t-5)\right\}$$

$$= \frac{3}{2}\{e^{-2t} - e^{-4t}\} + 2\{e^{-2t} \cdot e^{10} \cdot u(t-5) - e^{-4t} \cdot e^{20} \cdot u(t-5)\}$$

$$x = e^{-2t}\left\{\frac{3}{2} + 2e^{10} \cdot u(t-5)\right\} - e^{-4t}\left\{\frac{3}{2} + 2e^{20} \cdot u(t-5)\right\}$$

3 Determine (a) $L\{4 \cdot \delta(t-3)\}$, (b) $L\{e^{-3t} \cdot \delta(t-2)\}$.

4 Sketch the graph of $f(t) = 3 \cdot \delta(t) + 4 \cdot \delta(t-2) - 3 \cdot \delta(t-4)$ and determine its Laplace transform.

5 Solve the equation $\ddot{x} + 6\dot{x} + 10x = 7 \cdot \delta(t)$ given that, at $t = 0$, $x = -1$ and $\dot{x} = 0$.