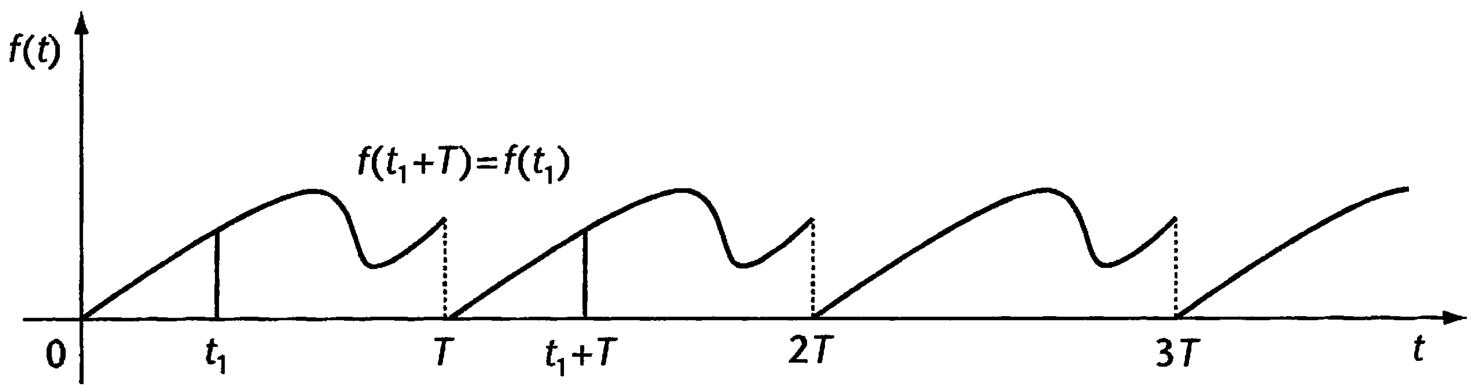
Transformasi Laplace Fungsi Periodik

Febrizal, MT

Fungsi Periodik

- Misalkan f(t) menyatakan sebuah fungsi periodik dengan periode T, jadi f(t+nT) = f(t) dengan bentuk grafik sebagai berikut:



- Jika kita deskripsikan periode pertama dengan $\bar{f}(t)$, maka;

$$\bar{f}(t) = \begin{cases} f(t) & \text{for } 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

- Periode kedua identik dengan periode pertama tetapi digeser sejauh T
- Oleh karena itu periode kedua bisa dinyataka dalam fungsi step sebagai:

$$\bar{f}(t-T)u(t-T) = \begin{cases} f(t) & \text{for } T \le t < 2T \\ 0 & \text{otherwise} \end{cases}$$

- Dengan mengacu pada hal tersebut, maka fungsi periodik f(t) bisa dinyatakan sebagai:

$$f(t) = \bar{f}(t)u(t) + \bar{f}(t-T)u(t-T) + \bar{f}(t-2T)u(t-2T) + \cdots$$

Because

$u(t)$ switches on $\bar{f}(t)$ at time $t = 0$, $u(t-T)$ switches on $\bar{f}(t-T)$ at time $t = T$ and $u(t-2T)$ switches on $\bar{f}(t-2T)$ at time $t = 2T$, etc.

- Sehingga transformasi laplace dari fungsi periodik tsb adalah:

$$L\{f(t)\} = L\{\bar{f}(t)u(t)\} + L\{\bar{f}(t-T)u(t-T)\} + L\{\bar{f}(t-2T)u(t-2T)\} + \cdots$$

$$L\{f(t)\} = \bar{F}(s) + e^{-sT}\bar{F}(s) + e^{-2sT}\bar{F}(s) + \cdots$$

$$L\{f(t)\} = \left(1 + e^{-sT} + e^{-2sT} + \ldots\right)\bar{F}(s)$$

$$\boxed{1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}}$$

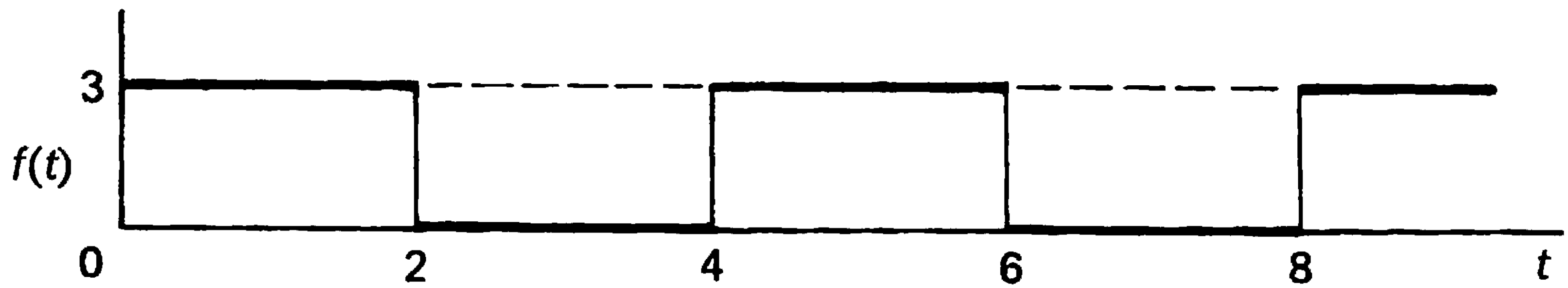
$$\boxed{1 + e^{-sT} + e^{-2sT} + \ldots = \frac{1}{1-e^{-sT}}}$$

$$L\{f(t)\} = \frac{1}{(1-e^{-sT})}\bar{F}(s) \text{ where } \bar{F}(s) = \int_0^T e^{-st} f(t)\, \mathrm{d}t$$

Example 1

Find the Laplace transform of the function $f(t)$ defined by

$$\left.\begin{aligned} f(t) &= 3 \quad 0 < t < 2 \\ &= 0 \quad 2 < t < 4 \end{aligned}\right\} \quad f(t+4) = f(t)$$



$$L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} \cdot f(t)\,dt$$

The function $f(t) = 3$ for $0 < t < 2$ and $f(t) = 0$ for $2 < t < 4$.

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^2 e^{-st} \cdot 3\,dt = \ldots\ldots\ldots\ldots$$

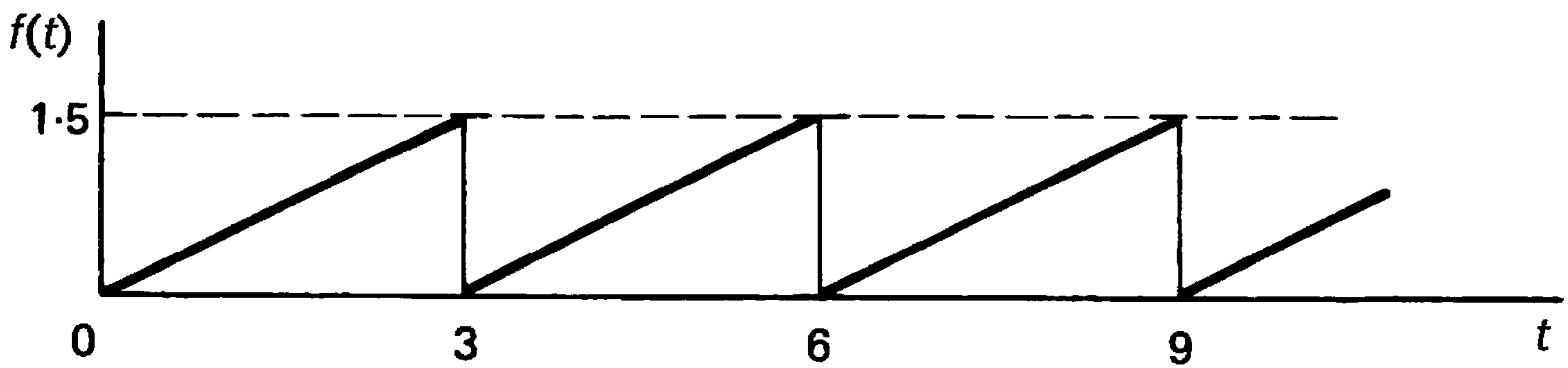
$$L\{f(t)\} = \frac{3}{1-e^{-4s}}\left[\frac{e^{-st}}{-s}\right]_0^2 = \frac{3}{1-e^{-4s}}\left\{\left(\frac{e^{-2s}}{-s}\right) - \left(\frac{1}{-s}\right)\right\}$$

$$= \frac{3}{1-e^{-4s}}\left\{\frac{1-e^{-2s}}{s}\right\} = \frac{3}{s(1+e^{-2s})}$$

Example 2

Find the Laplace transform of the periodic function defined by

$$f(t) = t/2 \qquad 0 < t < 3$$
$$f(t+3) = f(t)$$



Because in this case, period = 3, i.e. $T = 3$.

$$\therefore \; L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\, \mathrm{d}t$$

$$= \frac{1}{1 - e^{-3s}} \int_0^3 e^{-st} \cdot \left(\frac{t}{2}\right) \mathrm{d}t$$

$$\therefore \; 2(1 - e^{-3s}) L\{f(t)\} = \int_0^3 t \cdot e^{-st}\, \mathrm{d}t$$

Integrating by parts and simplifying the result gives

$$= \left[t \left(\frac{e^{-st}}{-s} \right) \right]_0^3 + \frac{1}{s} \int_0^3 e^{-st}\, \mathrm{d}t$$

$$= -\frac{3e^{-3s}}{s} + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^3$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}$$

$$\therefore \; L\{f(t)\} = \frac{1}{2s^2}\left\{1 - \frac{3se^{-3s}}{1 - e^{-3s}}\right\}$$

$$= \frac{1}{2s^2}\left\{1 - \frac{3s}{e^{3s} - 1}\right\}$$

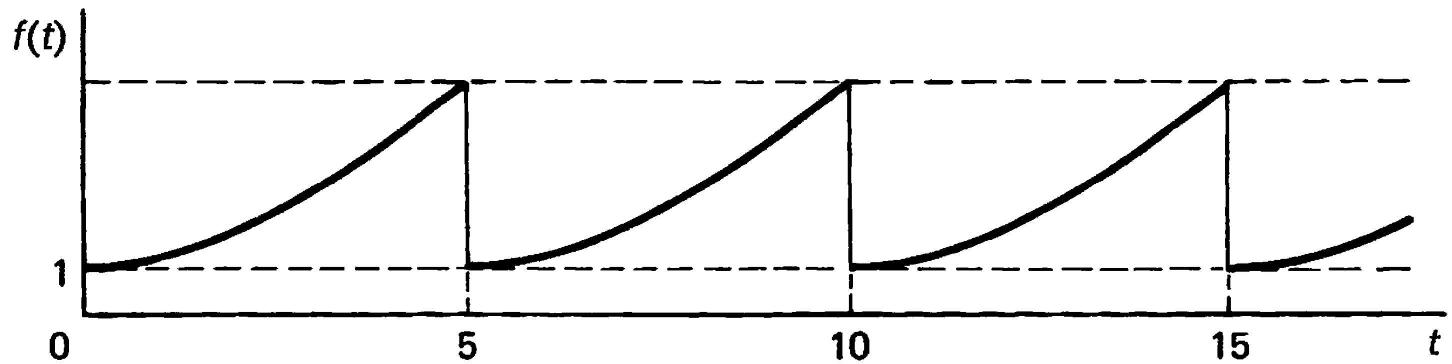
Example 3

Sketch the graph of the function

$$f(t) = e^t \qquad 0 < t < 5$$

$$f(t+5) = f(t)$$

and determine its Laplace transform.



Clearly, period = 5 $\quad \therefore\ T = 5$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\,\mathrm{d}t$$

$$L\{f(t)\} = \frac{1}{1-e^{-5s}} \int_0^5 e^{-st} \cdot e^{t}\,\mathrm{d}t$$

$$(1-e^{-5s})L\{f(t)\} = \int_0^5 e^{-(s-1)t}\,\mathrm{d}t$$

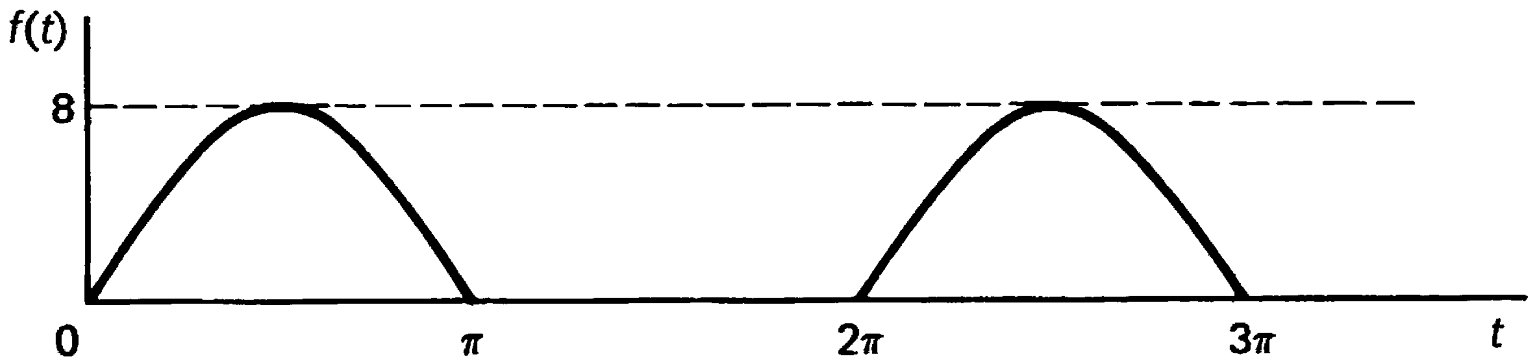
$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^5 = \frac{1}{s-1}\left\{1-e^{-5(s-1)}\right\}$$

$$\therefore\ L\{f(t)\} = \frac{1-e^{-5(s-1)}}{(s-1)(1-e^{-5s})}$$

Example 4

Determine the Laplace transform of the half-wave rectifier output waveform defined by

$$\left.\begin{aligned} f(t) &= 8\sin t \quad & 0 < t < \pi \\ &= 0 & \pi < t < 2\pi \end{aligned}\right\} \quad f(t+2\pi) = f(t)$$



Here the period is 2π i.e. $T = 2\pi$.

In general, for a periodic function of period T

So, for this example

$$L\{f(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} \cdot f(t)\,\mathrm{d}t$$

$$\therefore \quad (1 - e^{-2\pi s})L\{f(t)\} = \int_0^{\pi} e^{-st} \cdot 8 \sin t\,\mathrm{d}t$$

Writing $\sin t$ as the imaginary part of e^{jt}, i.e. $\sin t \equiv \mathscr{I} e^{jt}$,

$$(1 - e^{-2\pi s})L\{f(t)\} = 8\mathscr{I} \int_0^{\pi} e^{-st} \cdot e^{jt}\,\mathrm{d}t$$

$$= 8\mathscr{I} \int_0^{\pi} e^{-(s-j)t}\,\mathrm{d}t$$

and this you can finish off in the usual manner, giving

$$(1-e^{-2\pi s})L\{f(t)\} = 8\cdot\mathscr{I}\int_0^{\pi} e^{-(s-j)t}\,\mathrm{d}t$$

$$= 8\cdot\mathscr{I}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi}$$

$$= \mathscr{I}\left\{\frac{-8}{s-j}\left[e^{-(s-j)\pi}-1\right]\right\}$$

$$= 8\cdot\mathscr{I}\left\{\frac{1}{s-j}\left[1-e^{-s\pi}e^{j\pi}\right]\right\}$$

But $e^{j\pi} = \cos\pi + j\sin\pi = -1.$

$$\therefore\ (1-e^{-2\pi s})L\{f(t)\} = 8\cdot\mathscr{I}\left\{\frac{1}{s-j}(1+e^{-s\pi})\right\}$$

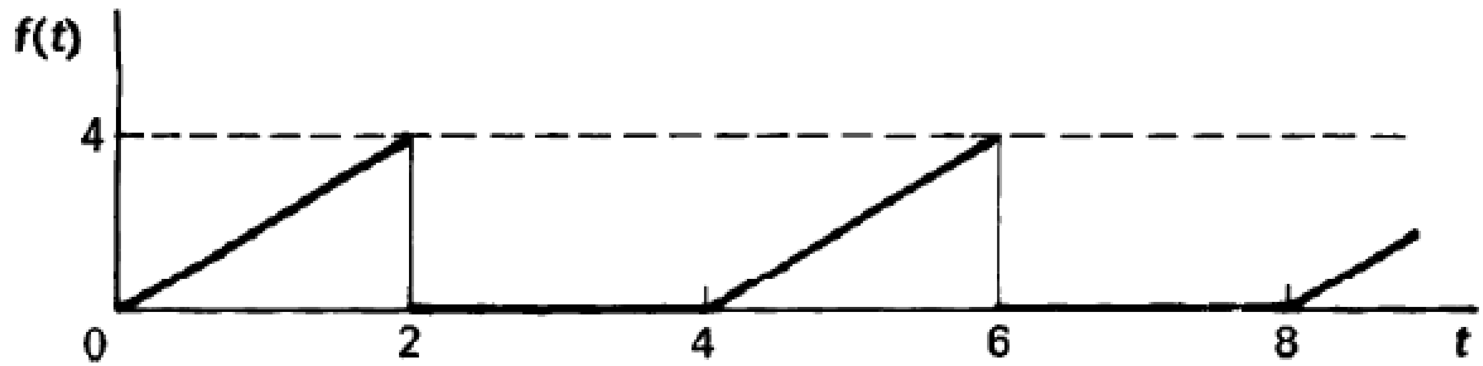
$$= 8\cdot\mathscr{I}\left\{\frac{s+j}{s^2+1}(1+e^{-\pi s})\right\} = 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$$

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\times 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$$

$$= \frac{8}{(1-e^{-\pi s})(s^2+1)}$$

Latihan

Determine the Laplace transform of the periodic function shown.



Invers Laplace Fungsi Periodik

- Menentukan invers laplace dari suatu fungsi periodik tidak bisa langsung seperti kasus sebelumnya, sebab transformasi laplace fungsi periodik diperoleh dari integral fungsi hanya satu perioda, bukan dari $t = 0$ sampai $t = \infty$
- Oleh karena itu kita tidak bisa menginverskan nya secara langsung.

- Contoh 1
  - Tentukan invers laplace dari fungsi berikut:

$$L^{-1}\left\{\frac{2+e^{-2s}-3e^{-s}}{s(1-e^{-2s})}\right\}$$

- Jawab
  - Hal pertama yang harus dilakukan adalah perhatikan bagian penyebutnya yaitu $(1-e^{-2s})$
  - Ini berarti fungsi nya adalah fungsi periodik dengan perioda 2
  - Langkah berikutnya adalah menuliskan $(1-e^{-2s})$ sbg penyebut menjadi $(1-e^{-2s})^{-1}$ sbg pembilang dan menyatakannya dalam deret binomial.

- Ingat bahwa: $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
- Sehingga:

$$\therefore\ (1-e^{-2s})^{-1} = 1 + (e^{-2s}) + (e^{-2s})^2 + (e^{-2s})^3 + \ldots$$

$$= 1 + e^{-2s} + e^{-4s} + e^{-6s} + \ldots$$

$$\therefore\ L\{f(t)\} = \frac{2 + e^{-2s} - 3e^{-s}}{s(1-e^{-2s})} = \frac{1}{s}(2 + e^{-2s} - 3e^{-s})(1-e^{-2s})^{-1}$$

$$= \frac{1}{s}(2 + e^{-2s} - 3e^{-s})(1 + e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + \ldots)$$

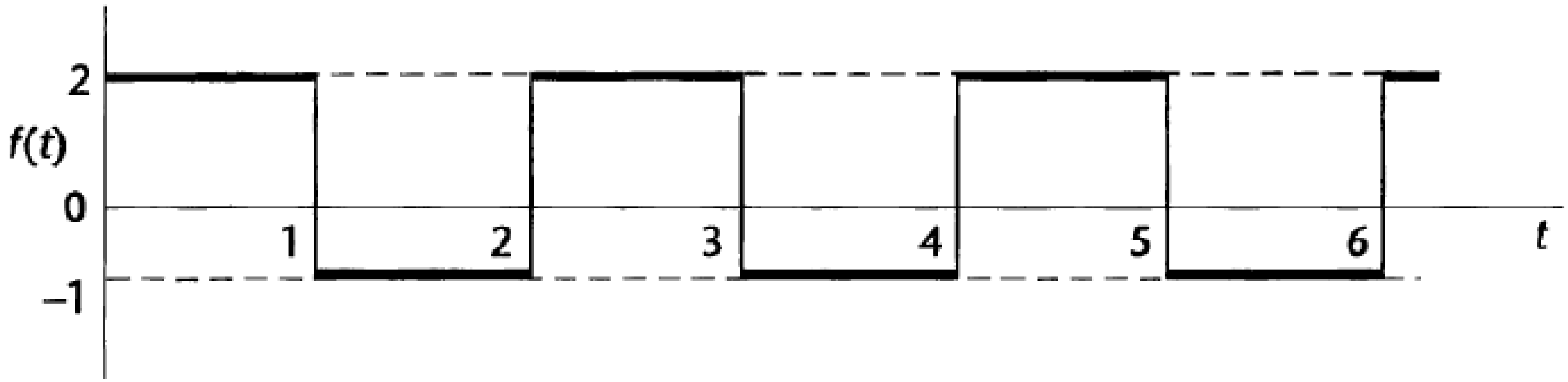
$$L\{f(t)\} = \frac{1}{s}\left\{\begin{matrix} 2 & & +2e^{-2s} & & +2e^{-4s} & & +2e^{-6s} & \ldots \\ & & +\ e^{-2s} & & +\ e^{-4s} & & +\ e^{-6s} & \ldots \\ & -3e^{-s} & & -3e^{-3s} & & -3e^{-5s} & & \ldots \end{matrix}\right\}$$

$$L\{f(t)\} = \frac{1}{s}\{2 - 3e^{-s} + 3e^{-2s} - 3e^{-3s} + 3e^{-4s} - 3e^{-5s} + \ldots\}$$

- Masing-masing bagian mempunyai bentuk $\frac{e^{-cs}}{s}$, sehingga bentuk f(t) nya adalah:

$$f(t) = 2u(t) - 3u(t-1) + 3u(t-2) - 3u(t-3) + 3u(t-4)\ldots$$

- Dari persamaan ini kita bisa gambarkan:



- Dan akhirnya kita bisa menyatakan fungsi periodik tsb sbg:

$$\left.\begin{aligned} f(t) &= 2 \quad & 0 < t < 1 \\ &= -1 \quad & 1 < t < 2 \end{aligned}\right\} \; f(t+2) = f(t)$$

Example 2

Determine $L^{-1}\left\{\dfrac{3(1-e^{-s})}{s(1-e^{-3s})}\right\}$ and sketch the resulting waveform of $f(t)$.

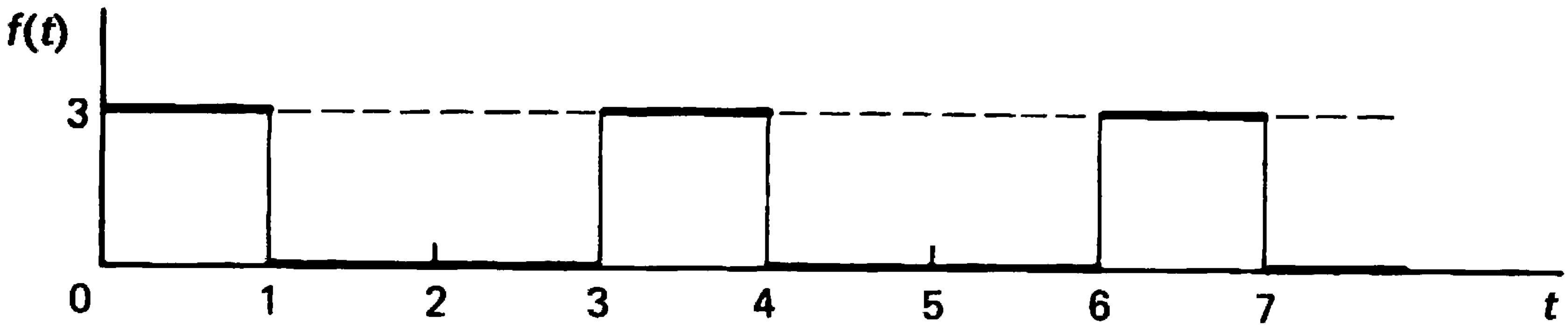
$$L\{f(t)\} = \frac{3}{s}(1-e^{-s})\left(1-e^{-3s}\right)^{-1}$$

$$L\{f(t)\} = \frac{3}{s}(1-e^{-s})\left(1+e^{-3s}+e^{-6s}+e^{-9s}+\ldots\right)$$

$$L\{f(t)\} = \frac{3}{s}(1-e^{-s}+e^{-3s}-e^{-4s}+e^{-6s}-e^{-7s}+\ldots)$$

$$= \frac{3}{s} - \frac{3e^{-s}}{s} + \frac{3e^{-3s}}{s} - \frac{3e^{-4s}}{s} + \frac{3e^{-6s}}{s} - \ldots$$

$$f(t) = 3u(t) - 3u(t-1) + 3u(t-3) - 3u(t-4) + \ldots$$



$$f(t) = 3 \quad 0 < t < 1 \atop f(t) = 0 \quad 1 < t < 3 \Bigg\} \; f(t+3) = f(t)$$

Example 3

If $L\{f(t)\} = \dfrac{1}{2s^2} - \dfrac{2e^{-4s}}{s(1 - e^{-4s})}$, determine $f(t)$ and sketch the waveform.

The first term is easy enough. In unit step form $L^{-1}\left\{\dfrac{1}{2s^2}\right\} = \dfrac{t}{2} \cdot u(t)$

From the second term

$$\frac{2e^{-4s}}{s(1-e^{-4s})}=\frac{2}{s}\left\{e^{-4s}\left(1-e^{-4s}\right)^{-1}\right\}$$

$$=\frac{2}{s}\{e^{-4s}(1+e^{-4s}+e^{-8s}+e^{-12s}+\ldots)\}$$

$$=\frac{2e^{-4s}}{s}+\frac{2e^{-8s}}{s}+\frac{2e^{-12s}}{s}+\frac{2e^{-16s}}{s}+\ldots$$

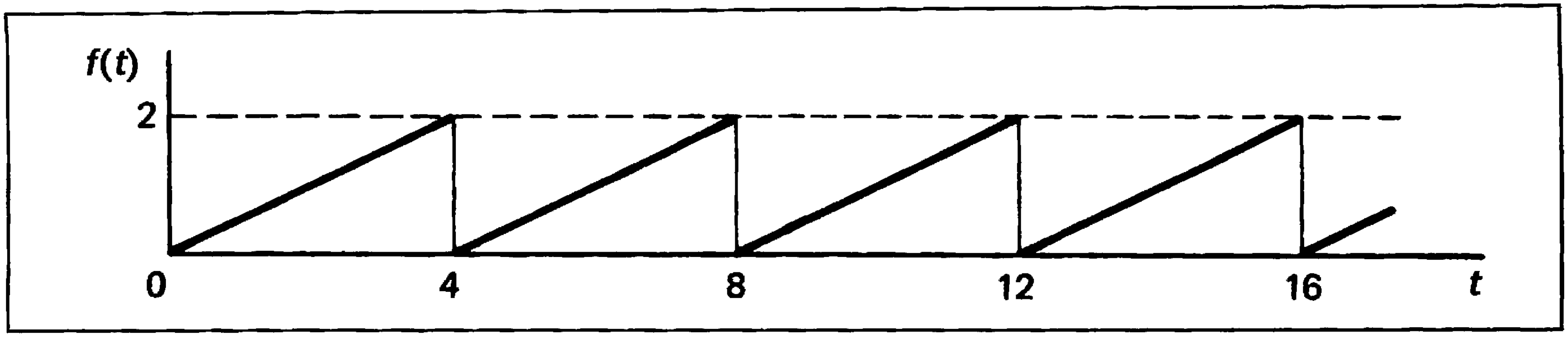
$$f(t)=\frac{t}{2}\cdot u(t)-2u(t-4)-2u(t-8)-2u(t-12)-\ldots$$

Now we have to draw the waveform. Consider the function terms up to each break point in turn.

$$0<t<4 \quad f(t)=\frac{t}{2} \qquad f(0)=0; \quad f(4)=2$$

$$4<t<8 \quad f(t)=\frac{t}{2}-2 \qquad f(4)=0; \quad f(8)=2$$

$$8<t<12 \quad f(t)=\frac{t}{2}-2-2 \qquad f(8)=0; \quad f(12)=2 \text{ etc.}$$



Expressed analytically, we finally have

$$f(t) = \frac{t}{2} \quad 0 < t < 4, \quad f(t+4) = f(t)$$

Latihan

- Tentukan $L^{-1}\left\{\dfrac{(1-e^{-2s})}{s(1-e^{-4s})}\right\}$ dan gambarkan grafik hasil dari fungsi f(t) tsb.