TRANSFORMASI LAPLACE FUNGSI STEP

BAGIAN II

FEBRIZAL, MT

The main points are

(a) $$\left.\begin{aligned} u(t-c) &= 0 \qquad 0 < t < c \\ &= 1 \qquad t \geq c \end{aligned}\right\}$$

(b) $$\left.\begin{aligned} L\{u(t-c)\} &= \frac{e^{-cs}}{s} \\ L\{u(t)\} &= \frac{1}{s} \end{aligned}\right\}$$

(c) $L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s)$ where $F(s) = L\{f(t)\}$

(d) If $F(s) = L\{f(t)\}$, then $e^{-cs}\cdot F(s) = L\{u(t-c)\}\cdot f(t-c)\}$

Now let us apply these to some further examples.

Example 1

Determine the expression $f(t)$ for which

$$L\{f(t)\} = \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}$$

We take each term in turn and find its inverse transform.

(a) $L^{-1}\left\{\frac{3}{s}\right\} = 3L^{-1}\left\{\frac{1}{s}\right\} = 3 \quad \text{i.e. } 3u(t)$

(b) $L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1)\cdot 4(t-1)$

(c) $L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = \ldots\ldots\ldots\ldots$

$$\boxed{u(t-2)\cdot 5(t-2)}$$

So we have $L^{-1}\left\{\dfrac{3}{s}\right\} = 3u(t)$

$$L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1)\cdot 4(t-1)$$

$$L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = u(t-2)\cdot 5(t-2)$$

$$\therefore\ F(t) = 3u(t) - u(t-1)\cdot 4(t-1) + u(t-2)\cdot 5(t-2)$$

To sketch the graph of $f(t)$ we consider the values of the function within the three sections $0 < t < 1$, $1 < t < 2$, and $2 < t$.

Between $t = 0$ and $t = 1$, $f(t) = 3$ ....

Because in this interval, $u(t) = 1$, but $u(t-1) = 0$ and $u(t-2) = 0$. In the same way, between $t = 1$ and $t = 2$, $f(t) = 7 - 4t$ .....

Because between $t = 1$ and $t = 2$, $u(t) = 1$, $u(t-1) = 1$, but $u(t-2) = 0$.

$$\therefore \; f(t) = 3 - 4(t-1) + 0 = 3 - 4t + 4 = 7 - 4t$$

Similarly, for $t > 2$, $f(t) = t - 3$ ......

Because for $t > 2$, $u(t) = 1$, $u(t-1) = 1$ and $u(t-2) = 1$

$$\therefore \; f(t) = 3 - 4(t-1) + 5(t-2)$$
$$= 3 - 4t + 4 + 5t - 10 = t - 3$$

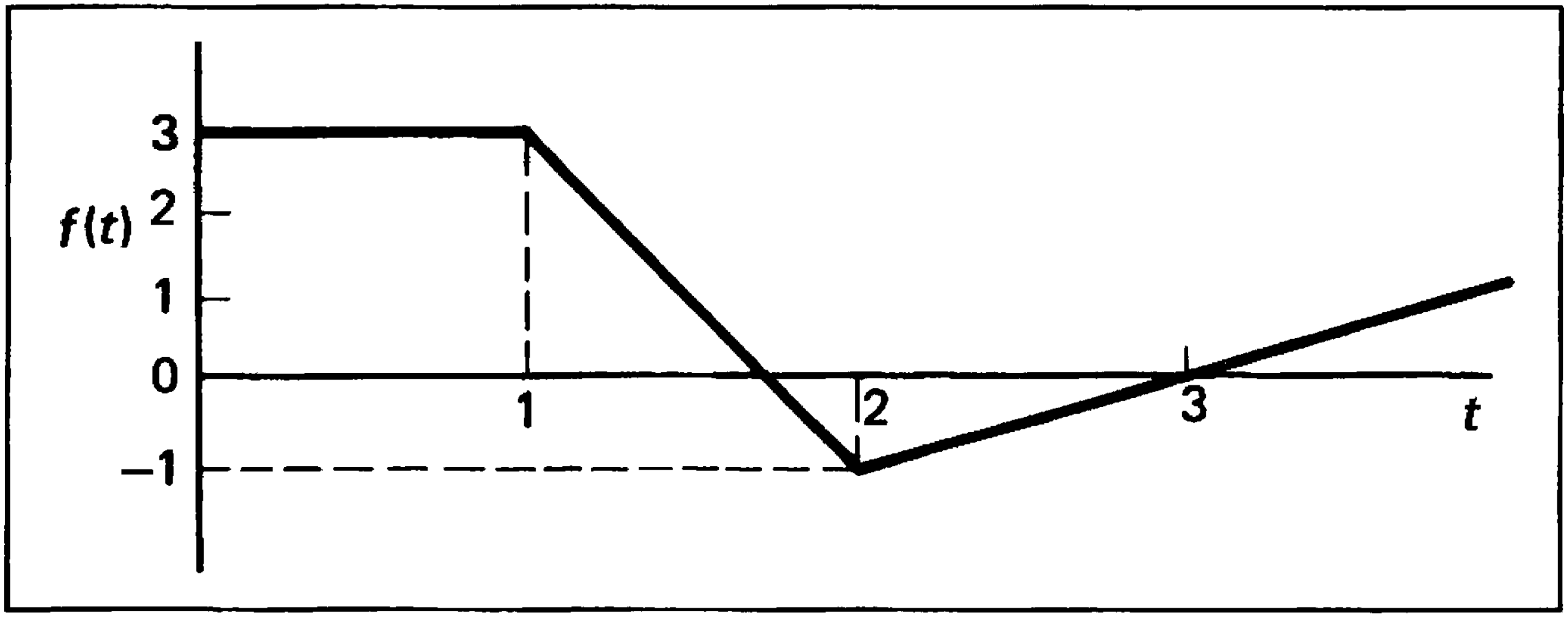
So, collecting the results together, we have

for $0 < t < 1$, $f(t) = 3$

$1 < t < 2$, $f(t) = 7 - 4t$ $(t = 1, f(t) = 3;\ t = 2, f(t) = -1)$

$2 < t$, $f(t) = t - 3$ $(t = 2, f(t) = -1;\ t = 3, f(t) = 0)$

Using these facts we can sketch the graph of $f(t)$, which is



Example 2

Determine the expression $f(t) = L^{-1}\left\{\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right\}$ and sketch the graph of $f(t)$.

First we express the inverse transform of each term in terms of the unit step function.

$$L^{-1}\left\{\frac{2}{s}\right\} = 2u(t); \quad L^{-1}\left\{\frac{3e^{-s}}{s^2}\right\} = u(t-1)\cdot 3(t-1)$$

$$L^{-1}\left\{\frac{3e^{-3s}}{s^2}\right\} = u(t-3)\cdot 3(t-3)$$

$$\therefore \quad f(t) = 2u(t) + u(t-1)\cdot 3(t-1) - u(t-3)\cdot 3(t-3)$$

So there are 'break points', i.e. changes of function, at $t = 1$ and $t = 3$, and we investigate $f(t)$ within the three intervals.

$0 < t < 1 \qquad f(t) = \ldots\ldots\ldots\ldots$

$1 < t < 3 \qquad f(t) = \ldots\ldots\ldots\ldots$

$3 < t \qquad\quad f(t) = \ldots\ldots\ldots\ldots$

$$0 < t < 1,\ f(t) = 2; \quad 1 < t < 3,\ f(t) = 3t - 1; \quad 3 < t,\ f(t) = 8$$

Because with

$0 < t < 1$, $u(t) = 1$, but $u(t-1) = u(t-3) = 0$ $\quad \therefore\ f(t) = 2$

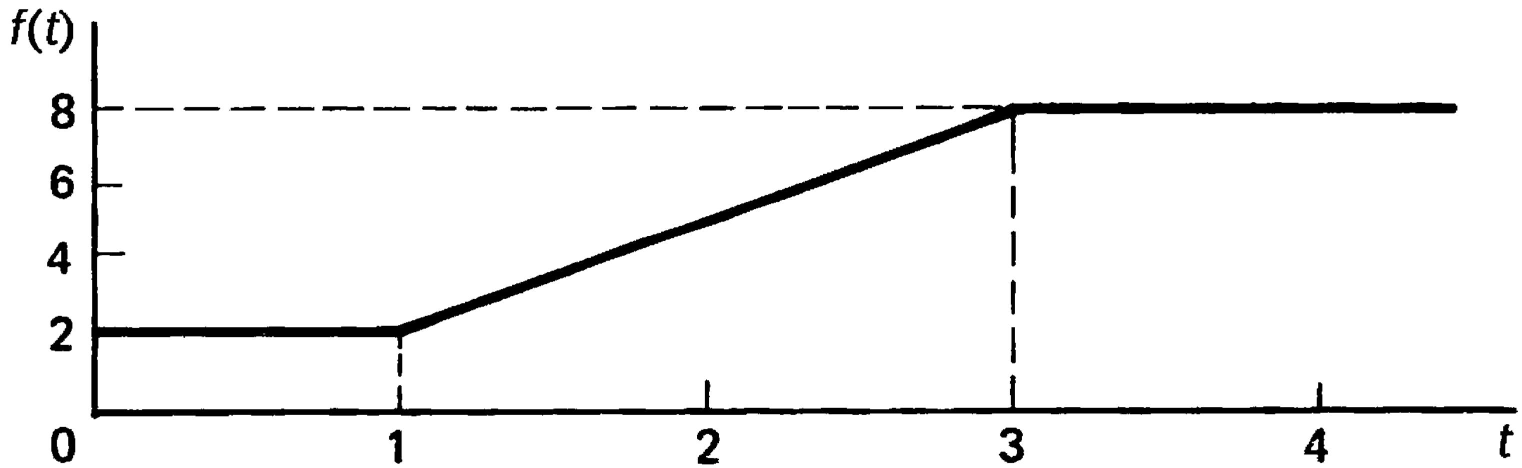
$1 < t < 3$, $u(t) = 1$, $u(t-1) = 1$, but $u(t-3) = 0$

$\therefore\ f(t) = 2 + 3(t-1) = 3t - 1$ $\quad \therefore\ f(t) = 3t - 1$

$3 < t$, $u(t) = 1$, $u(t-1) = 1$, $u(t-3) = 1$

$\therefore\ f(t) = 2 + 3t - 3 - 3t + 9$ $\quad \therefore\ f(t) = 8$

Therefore, the graph of $f(t)$ is ....



Example 3

If $f(t) = L^{-1}\left\{\frac{(1-e^{-2s})(1+e^{-4s})}{s^2}\right\}$, determine $f(t)$ and sketch the graph of the function.

Although at first sight this looks more complicated, we simply multiply out the numerator and proceed as before.

$$f(t) = L^{-1}\left\{\frac{1-e^{-2s}+e^{-4s}-e^{-6s}}{s^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}-\frac{e^{-2s}}{s^2}+\frac{e^{-4s}}{s^2}-\frac{e^{-6s}}{s^2}\right\}$$

We now write down the inverse transform of each term in terms of the unit function, so that

$$\boxed{f(t) = u(t)\cdot t - u(t-2)\cdot(t-2) + u(t-4)\cdot(t-4) - u(t-6)\cdot(t-6)}$$

and we can see there are break points at $t = 2$, $t = 4$, $t = 6$.

For $0 < t < 2$, $f(t) = t - 0 + 0 - 0$ $\quad f(t) = t$

$2 < t < 4$, $f(t) = t - (t - 2) + 0 - 0$ $\quad f(t) = 2$

$4 < t < 6$, $f(t) = t - (t - 2) + (t - 4) - 0$ $\quad f(t) = t - 2$

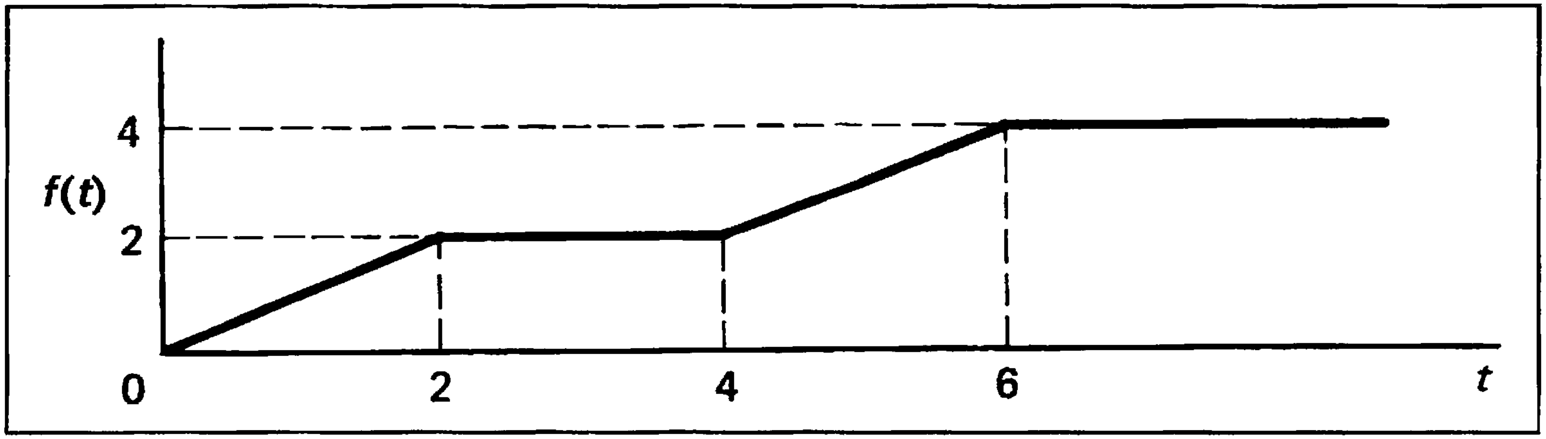
$6 < t$, $f(t) = t - (t - 2) + (t - 4) - (t - 6)$ $\quad f(t) = 4$

The second and fourth components are constant, but before sketching the graph of the function, we check the values of $f(t) = t$ and $f(t) = t - 2$ at the relevant break points.

$f(t) = t$. At $t = 0$, $f(t) = 0$; at $t = 2$, $f(t) = 2$

$f(t) = t - 2$. At $t = 4$, $f(t) = 2$; at $t = 6$, $f(t) = 4$.

So the graph of the function is ....



It is always wise to calculate the function values at break points, since discontinuities, or jumps, sometimes occur.

LATIHAN

If $f(t) = L^{-1}\left\{\dfrac{(1-e^{-s})(1+e^{-2s})}{s^2}\right\}$, find $f(t)$ in terms of the unit step function.

Now for one in reverse.

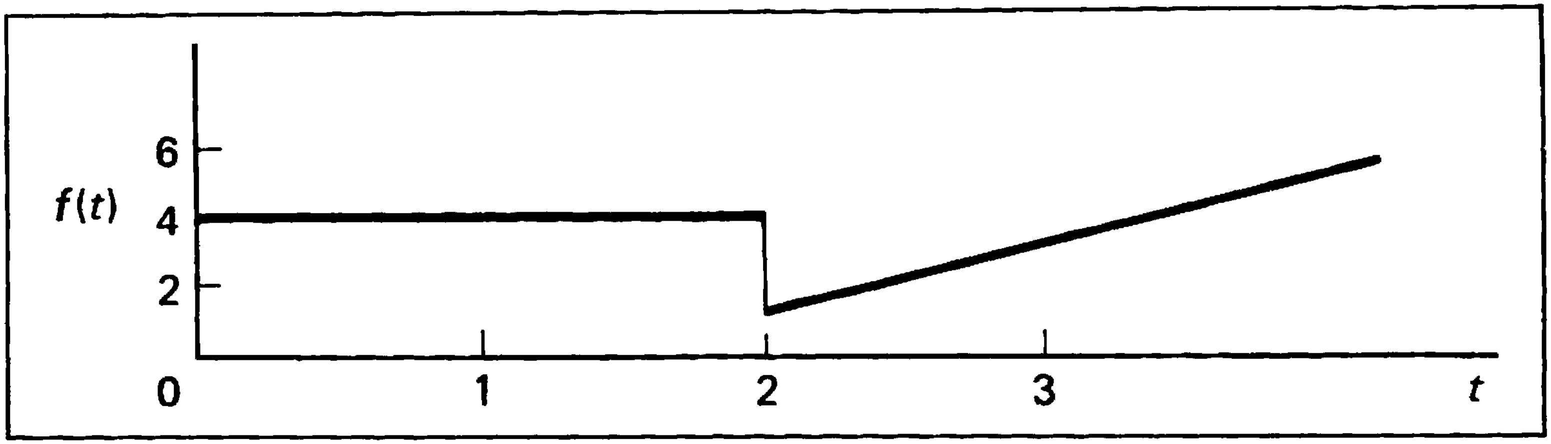
Example 4

A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= 4 \qquad &\text{for } 0 < t < 2 \\ &= 2t - 3 \qquad &\text{for } 2 < t. \end{aligned}$$

Sketch the graph of the function and determine its Laplace transform.

We see that for $t = 0$ to $t = 2$, $f(t) = 4$.



Notice the discontinuity at $t = 2$.

Expressing the function in unit step form:

$$f(t) = 4u(t) - 4u(t-2) + u(t-2) \cdot (2t-3)$$

Note that the second term cancels $f(t) = 4$ at $t = 2$ and that the third switches on $f(t) = 2t - 3$ at $t = 2$.

Before we can express this in Laplace transforms, $(2t - 3)$ in the third term must be written as a function of $(t - 2)$ to correspond to $u(t - 2)$. Therefore, we write $2t - 3$ as $2(t - 2) + 1$.

$$
\begin{aligned}
\text{Then } f(t) &= 4u(t) - 4u(t-2) + u(t-2) \cdot \{2(t-2) + 1\} \\
&= 4u(t) - 4u(t-2) + u(t-2) \cdot 2(t-2) + u(t-2) \\
&= 4u(t) - 3u(t-2) + u(t-2) \cdot 2(t-2)
\end{aligned}
$$

$$\therefore\ L\{f(t)\} = \ldots\ldots\ldots\ldots$$

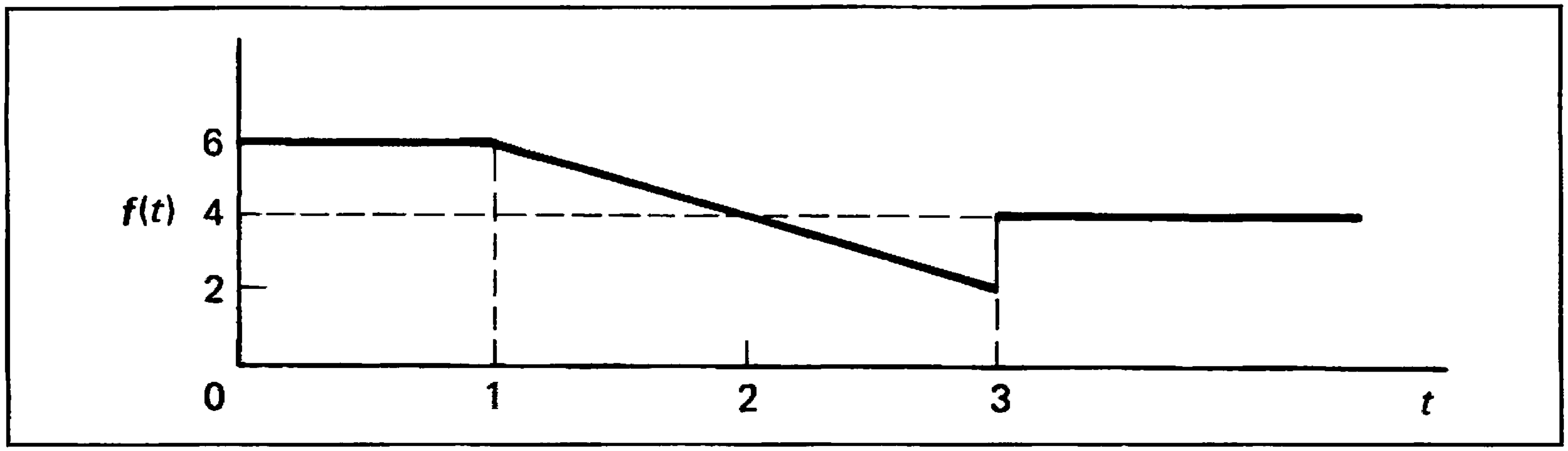
$$\boxed{L\{f(t)\} = \frac{4}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-2s}}{s^2}}$$

Example 5

A function is defined by

$$
\begin{aligned}
f(t) &= 6 & 0 < t < 1 \\
&= 8 - 2t & 1 < t < 3 \\
&= 4 & 3 < t.
\end{aligned}
$$

Sketch the graph and find the Laplace transform of the function.



Expressing this in unit step form we have

$$f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot(8-2t)$$
$$\quad - u(t-3)\cdot(8-2t) + u(t-3)\cdot 4$$

where the second term switches off the first function $f(t) = 6$ at $t = 1$ and the third term switches on the second function $f(t) = 8 - 2t$, which in turn is switched off by the fourth term at $t = 3$ and replaced by $f(t) = 4$ in the fifth term.

Before we can write down the transforms of the third and fourth terms, we must express $f(t) = 8 - 2t$ in terms of $(t-1)$ and $(t-3)$ respectively.

$$8 - 2t = 6 + 2 - 2t = 6 - 2(t-1)$$
$$8 - 2t = 2 + 6 - 2t = 2 - 2(t-3)$$

$$\begin{aligned}\therefore\ f(t) &= 6u(t) - 6u(t-1) + u(t-1)\cdot\{6 - 2(t-1)\}\\ &\quad - u(t-3)\cdot\{2 - 2(t-3)\} + 4u(t-3)\\ &= 6u(t) - 6u(t-1) + 6u(t-1)\\ &\quad - u(t-1)\cdot 2(t-1) - 2u(t-3)\\ &\quad + u(t-3)\cdot 2(t-3) + 4u(t-3)\end{aligned}$$

which simplifies finally to $f(t) = \dots\dots\dots\dots$

$$f(t) = 6u(t) - u(t-1)\cdot 2(t-1) + u(t-3)\cdot 2(t-3) + 2u(t-3)$$

from which $L\{f(t)\} = \dots\dots\dots\dots$

$$L\{f(t)\} = \frac{6}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} + \frac{2e^{-3s}}{s}$$

LATIHAN

A function $f(t)$ is defined by

$$f(t) = 4 \qquad 0 < t < 3$$
$$= 2t + 1 \qquad 3 < t.$$

Sketch the graph of the function and determine its Laplace transform.

Note that, in building up the function in unit step form

(a) to 'switch on' a function $f(t)$ at $t = c$, we add the term $u(t-c) \cdot f(t-c)$

(b) to 'switch off' a function $f(t)$ at $t = c$, we subtract $u(t-c) \cdot f(t-c)$.

QUIS II

A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= t^2 && 0 < t < 2 \\ &= 4 && 2 < t < 5 \\ &= 0 && 5 < t. \end{aligned}$$

Determine (a) the function in terms of the unit step function

(b) the Laplace transform of $f(t)$.