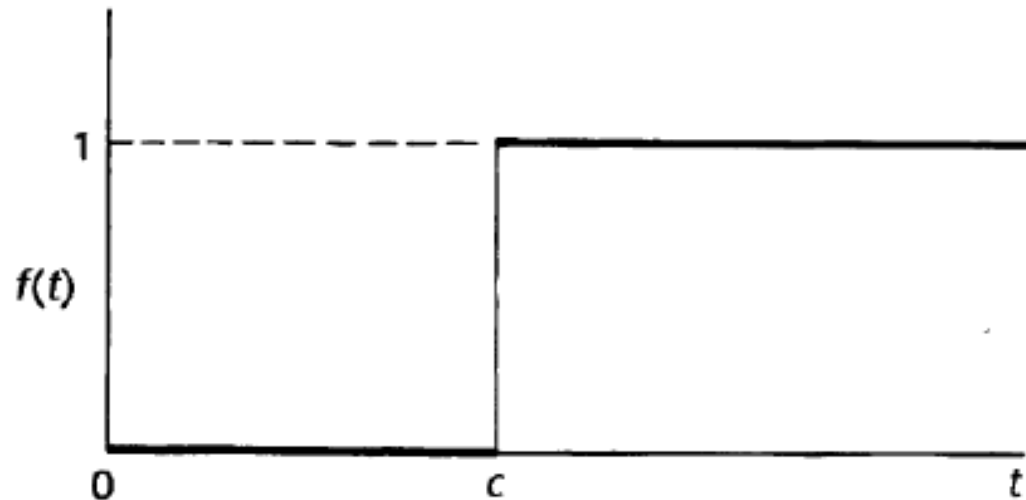


Transformasi Laplace Fungsi Step

Febrizal, MT

Pengertian Fungsi Step

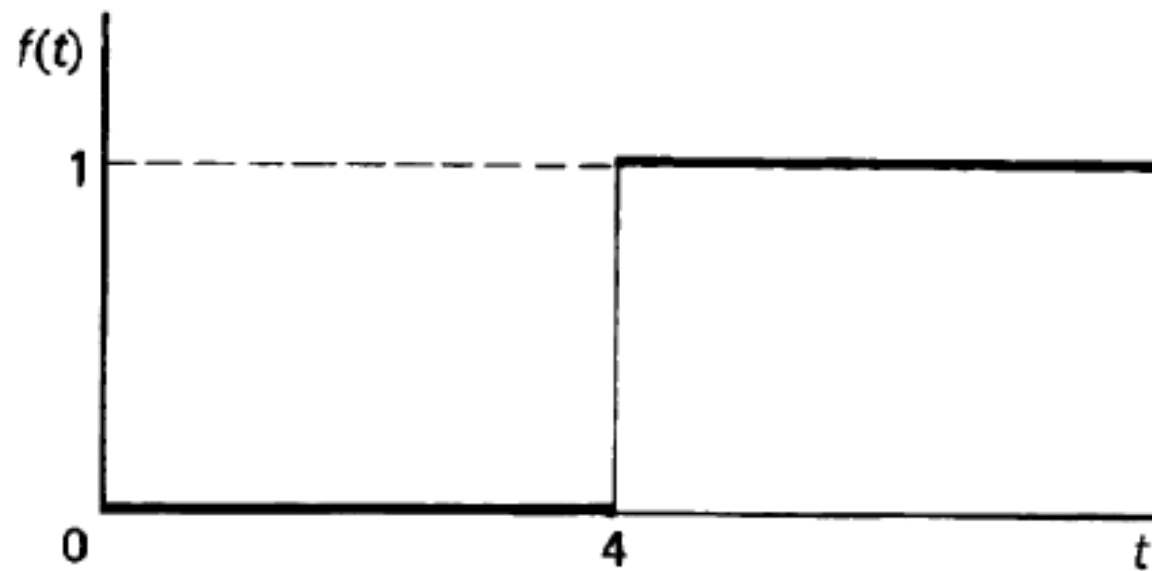
- Fungsi step adalah fungsi yang bernilai nol pada saat $t < c$ dan bernilai 1 pada saat $t \geq c$



$$f(t) = 0 \quad \text{for } t < c$$
$$f(t) = 1 \quad \text{for } t \geq c$$

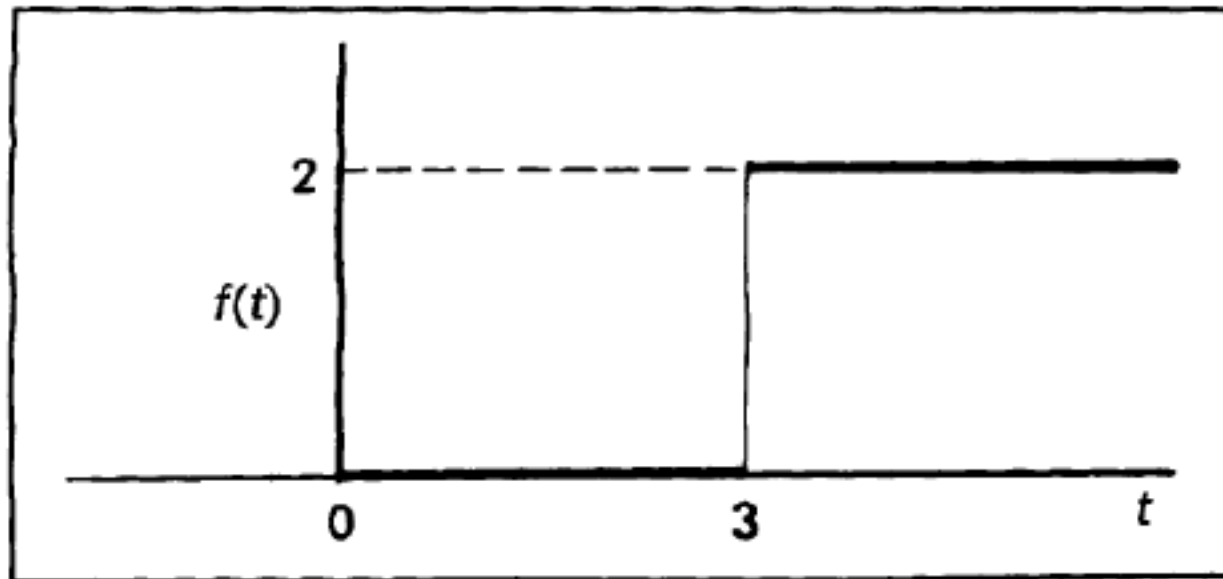
- Fungsi ini dinotasikan dengan $f(t) = u(t - c)$
- Dimana c merupakan nilai t yang menandai perubahan nilai fungsi dari 0 ke 1

- Dengan demikian, maka fungsi berikut dinyatakan sebagai..



$$f(t) = u(t - 4)$$

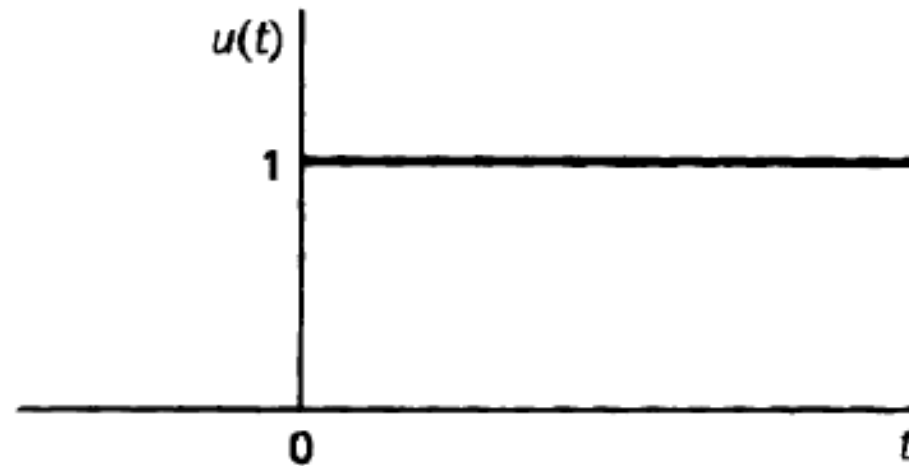
- Serupa dengan diatas, grafik dari:
- $f(t) = 2u(t - 3)$ adalah...



Berarti fungsi step $u(t - c)$ hanya mempunyai dua nilai
yaitu: untuk $t < c$, $u(t - c) = 0$
dan $t \geq c$, $u(t - c) = 1$

Fungsi step pada titik awal

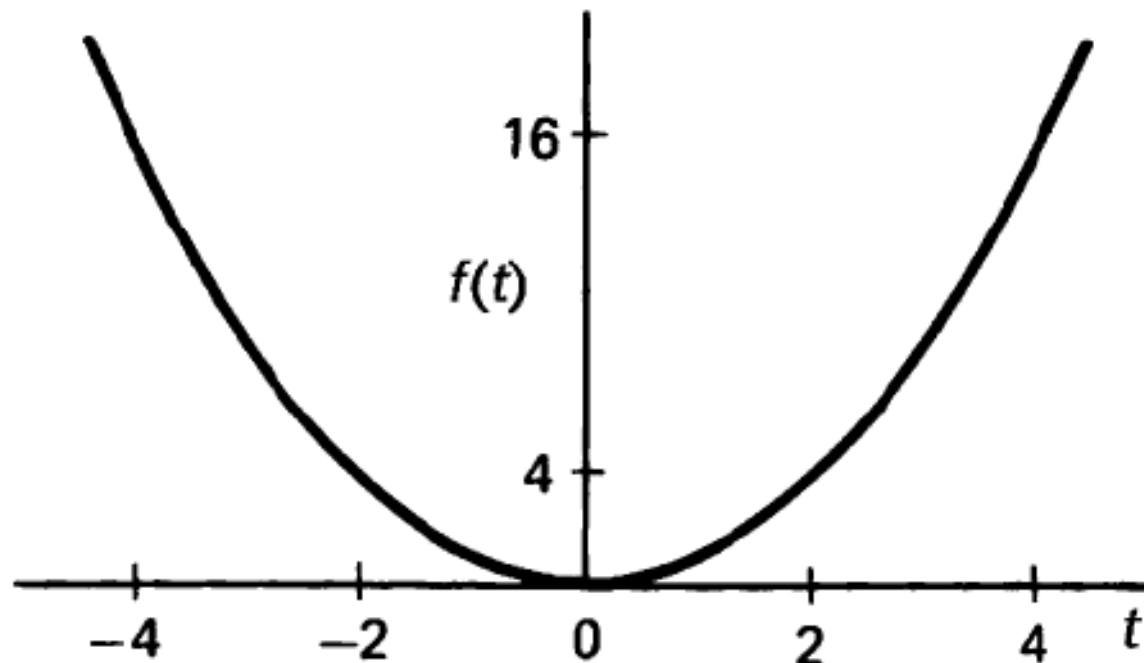
- Jika fungsi step terjadi pada titik awal (titik 0), berarti $c = 0$, maka fungsi $f(t) = u(t - c)$ menjadi $f(t) = u(t)$



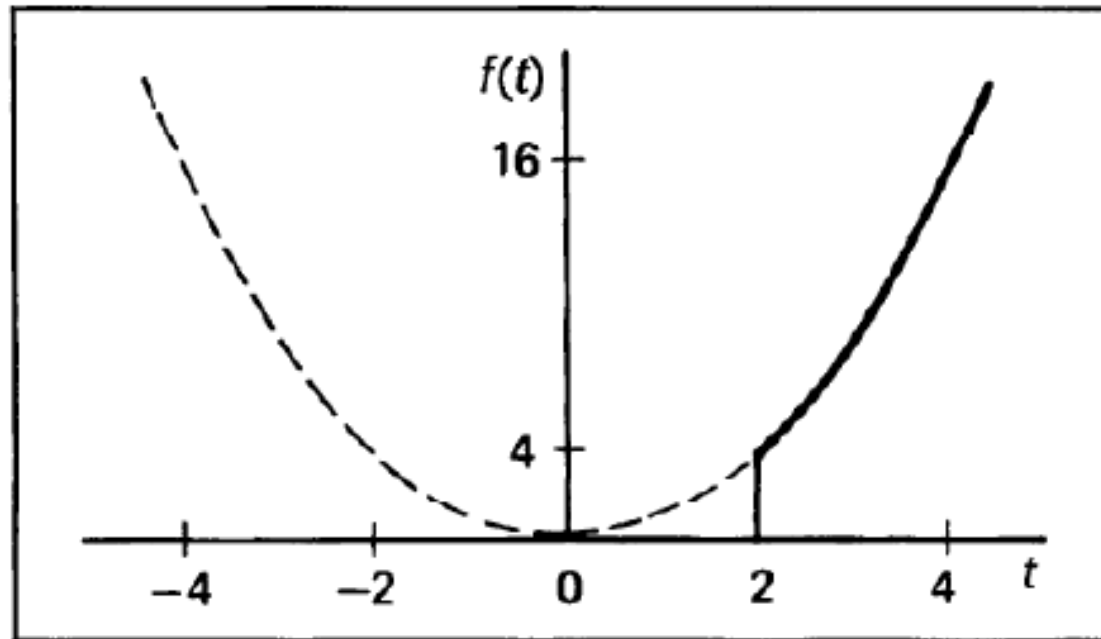
i.e. $u(t) = 0$ for $t < 0$
 $u(t) = 1$ for $t \geq 0$.

Pengaruh dari Fungsi Step

- Berikut ini adalah fungsi $f(t) = t^2$



- Grafik dari $f(t) = u(t-2) \cdot t^2$ adalah...



For $t < 2$, $u(t-2) = 0 \quad \therefore u(t-2) \cdot t^2 = 0 \cdot t^2 = 0$

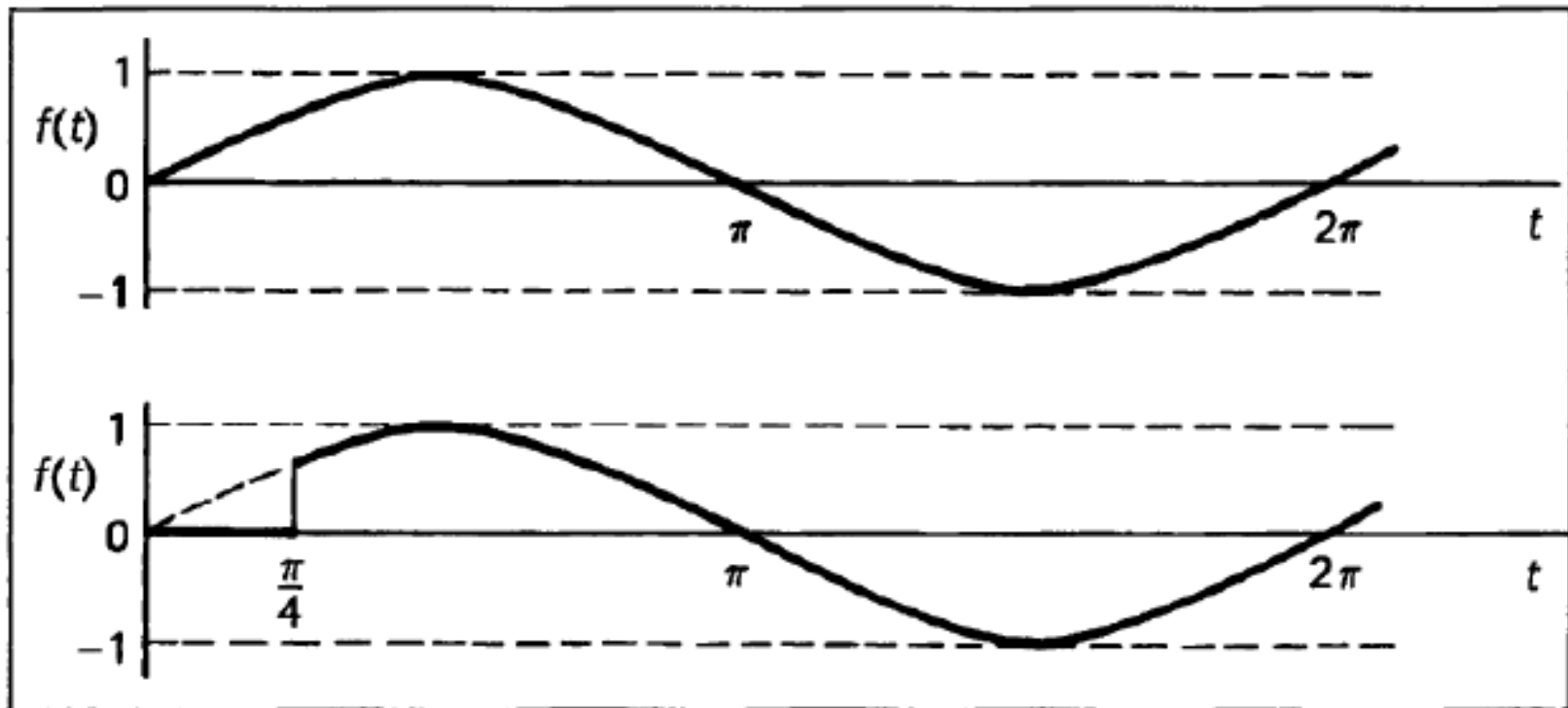
$t \geq 2$, $u(t-2) = 1 \quad \therefore u(t-2) \cdot t^2 = 1 \cdot t^2 = t^2$

- Berarti fungsi $u(t-2)$ menghilangkan fungsi t^2 utk semua nilai t sampai $t = 2$ dan mengembalikan fungsi t^2 pada $t = 2$

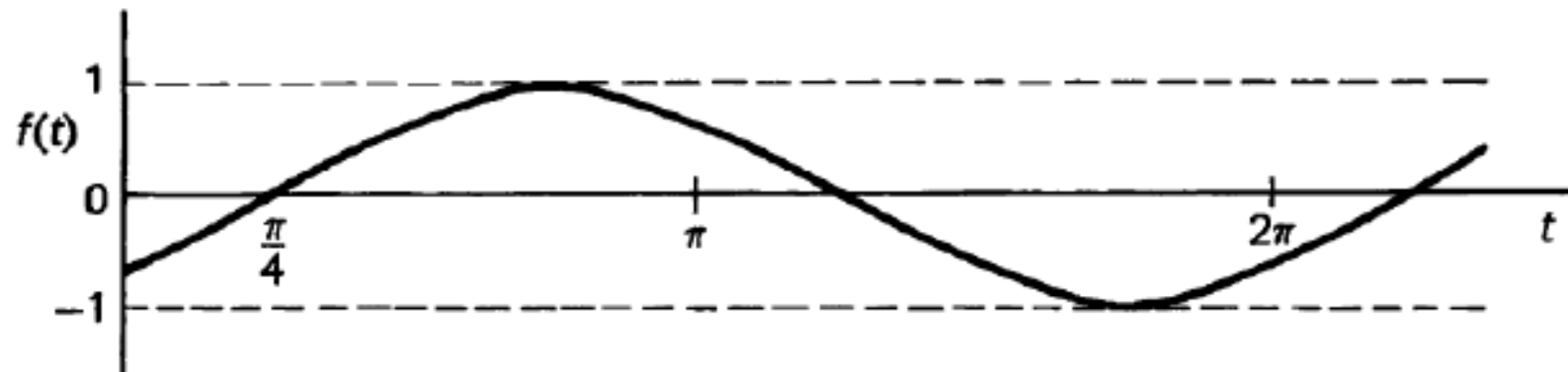
- Sekarang coba gambarkan grafik fungsi:

(a) $f(t) = \sin t$ for $0 < t < 2\pi$

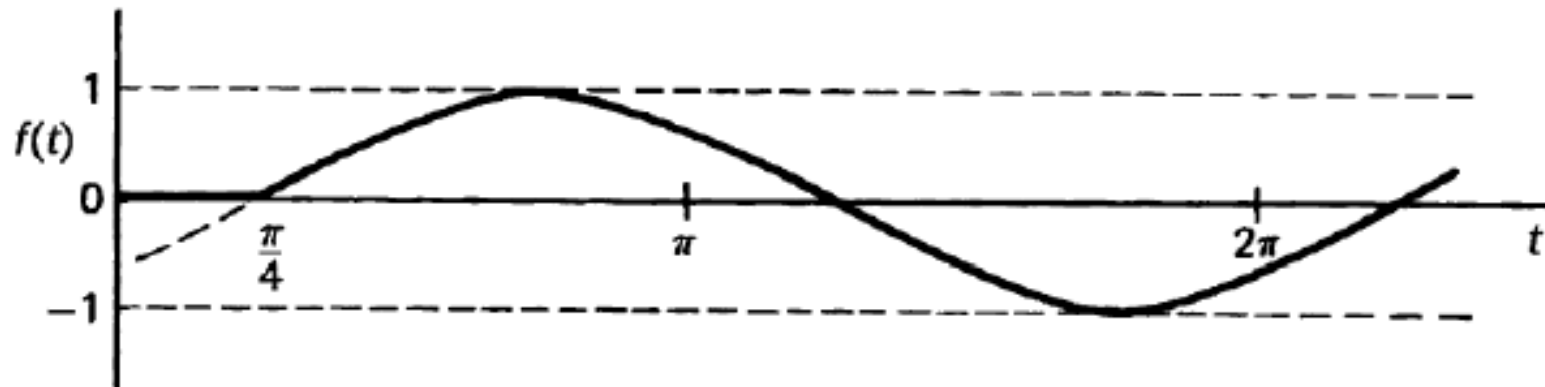
(b) $f(t) = u(t - \pi/4) \cdot \sin t$ for $0 < t < 2\pi$.



- Gambarkan grafik $f(t) = \sin(t - \pi/4)$ adalah sbb:

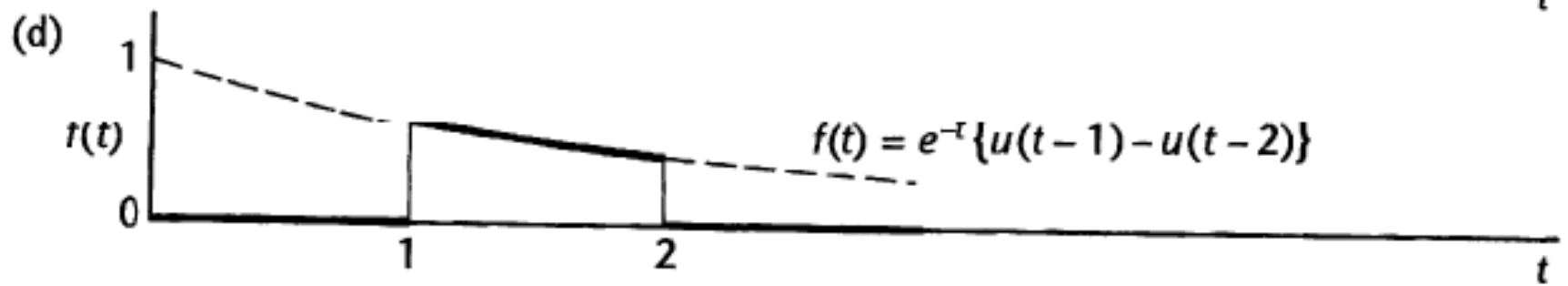
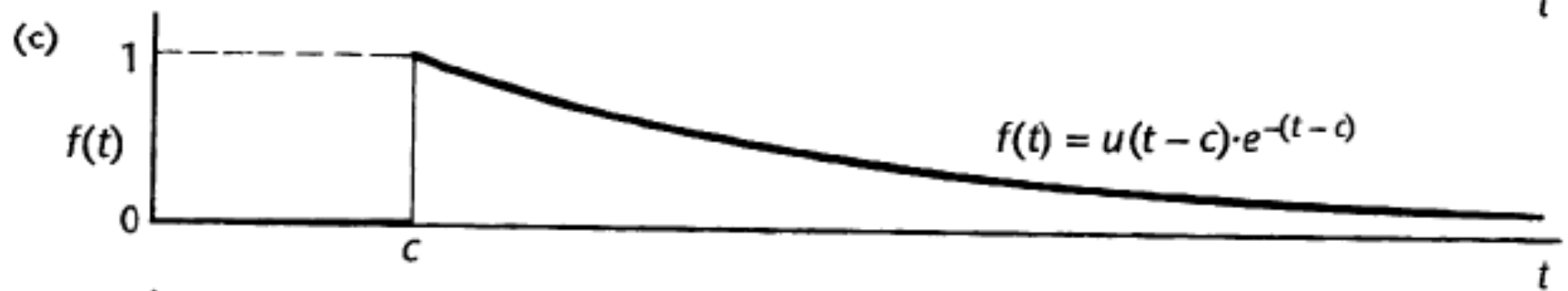
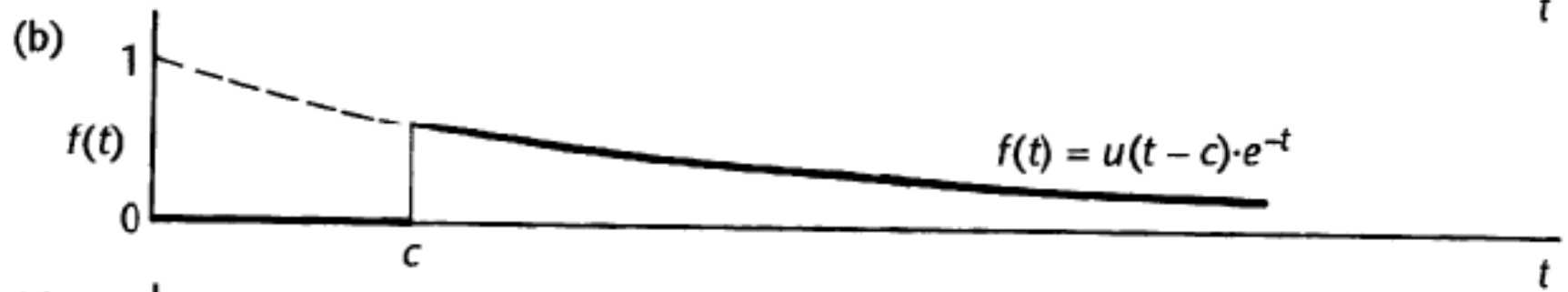
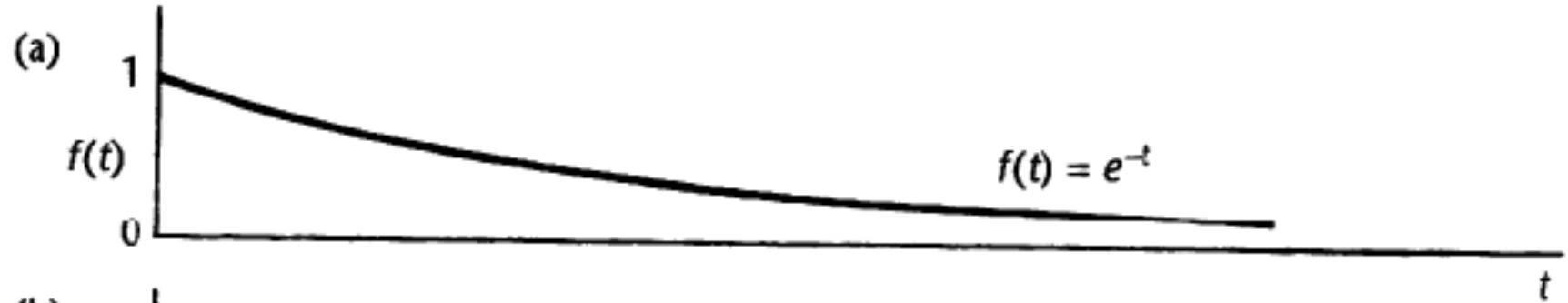


- Sedangkan grafik $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ adlh:



- Grafik tsb jg merupakan grafik $f(t) = \sin t$ utk $t > 0$ yg digeser sejauh $\pi/4$

- Secara umum, grafik $f(t) = u(t-c)$. Sin $(t-c)$ adalah grafik $f(t) = \sin t$ ($t > 0$) yang digeser sepanjang sumbu t sejauh c .
- Dgn aturan tsb, untuk $t > 0$ gambarkan grafik dari:
 - (a) $f(t) = e^{-t}$
 - (b) $f(t) = u(t - c) \cdot e^{-t}$
 - (c) $f(t) = u(t - c) \cdot e^{-(t-c)}$
 - (d) $f(t) = e^{-t} \{u(t - 1) - u(t - 2)\}$.



Transformasi Laplace dari $u(t-c)$

$$L\{u(t-c)\} = \int_0^{\infty} e^{-st} u(t-c) dt$$

but

$$e^{-st} u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$\begin{aligned} L\{u(t-c)\} &= \int_0^{\infty} e^{-st} u(t-c) dt = \int_c^{\infty} e^{-st} dt \\ &= \left[\frac{e^{-st}}{-s} \right]_c^{\infty} = \frac{e^{-sc}}{s} \end{aligned}$$

- Sehingga transformasi laplace dari fungsi step pada titik awal adalah:

$$L\{u(t)\} = \dots\dots\dots$$

Because $c = 0$.

So
$$L\{u(t - c)\} = \frac{e^{-cs}}{s}$$

and
$$L\{u(t)\} = \frac{1}{s}.$$

Laplace transform of $u(t - c) \cdot f(t - c)$ (the second shift theorem)

Because

$$L\{u(t - c) \cdot f(t - c)\} = \int_0^{\infty} e^{-st} u(t - c) \cdot f(t - c) dt$$

$$\text{but } e^{-st} u(t - c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t - c) \cdot f(t - c)\} = \int_c^{\infty} e^{-st} f(t - c) dt$$

$$L\{u(t - c) \cdot (f(t - c))\} = e^{-cs} L\{f(t)\} = e^{-cs} F(s)$$

$$\begin{aligned}\text{So } L\{u(t-4) \cdot (t-4)^2\} &= e^{-4s} \cdot F(s) \quad \text{where } F(s) = L\{t^2\} \\ &= e^{-4s} \left(\frac{2!}{s^3}\right) = \frac{2e^{-4s}}{s^3}\end{aligned}$$

Note that $F(s)$ is the transform of t^2 and *not* of $(t-4)^2$.

In the same way:

$$\begin{aligned}L\{u(t-3) \cdot \sin(t-3)\} &= e^{-3s} \cdot F(s) \quad \text{where } F(s) = L\{\sin t\} \\ &= \frac{1}{s^2 + 1}\end{aligned}$$

$$L\{u(t-3) \cdot \sin(t-3)\} = e^{-3s} \left(\frac{1}{s^2 + 1}\right)$$

Latihan

So now do these in the same way.

$$(a) \quad L\{u(t-2) \cdot (t-2)^3\} = \dots\dots\dots$$

$$(b) \quad L\{u(t-1) \cdot \sin 3(t-1)\} = \dots\dots\dots$$

$$(c) \quad L\{u(t-5) \cdot e^{(t-5)}\} = \dots\dots\dots$$

$$(d) \quad L\{u(t-\pi/2) \cdot \cos 2(t-\pi/2)\} = \dots\dots\dots$$

Here they are

$$\begin{aligned} \text{(a)} \quad L\{u(t-2) \cdot (t-2)^3\} &= e^{-2s} \cdot F(s) \quad \text{where } F(s) = L\{t^3\} \\ &= e^{-2s} \left(\frac{3!}{s^4} \right) = \frac{6e^{-2s}}{s^4} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad L\{u(t-1) \cdot \sin 3(t-1)\} &= e^{-s} \cdot F(s) \quad \text{where } F(s) = L\{\sin 3t\} \\ &= e^{-s} \left(\frac{3}{s^2+9} \right) = \frac{3e^{-s}}{s^2+9} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad L\{u(t-5) \cdot e^{(t-5)}\} &= e^{-5s} \cdot F(s) \quad \text{where } F(s) = L\{e^t\} \\ &= e^{-5s} \left(\frac{1}{s-1} \right) = \frac{e^{-5s}}{s-1} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad L\{u(t-\pi/2) \cdot \cos 2(t-\pi/2)\} &= e^{-\pi s/2} \cdot F(s) \quad \text{where } F(s) = L\{\cos 2t\} \\ &= e^{-\pi s/2} \left(\frac{s}{s^2+4} \right) = \frac{s \cdot e^{-\pi s/2}}{s^2+4} \end{aligned}$$

So $L\{u(t - c) \cdot f(t - c)\} = e^{-cs} \cdot F(s)$

where $F(s) = L\{f(t)\}$.

Written in reverse, this becomes

If $F(s) = L\{f(t)\}$,

then $e^{-cs} \cdot F(s) = L\{u(t - c) \cdot f(t - c)\}$

where c is real and positive.

This is known as the *second shift theorem*.

Example 1

Find the function whose transform is $\frac{e^{-4s}}{s^2}$.

The numerator corresponds to e^{-cs} where $c = 4$ and therefore indicates $u(t - 4)$.

Then $\frac{1}{s^2} = F(s) = L\{t\} \quad \therefore f(t) = t$.

$$\therefore L^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} = u(t - 4) \cdot (t - 4)$$

Example 2

Determine $L^{-1}\left\{\frac{6e^{-2s}}{s^2 + 4}\right\}$.

The numerator contains e^{-2s} and therefore indicates $u(t - 2)$.

$\frac{6}{s^2 + 4}$, can be written as $3\left(\frac{2}{s^2 + 4}\right)$

$$\therefore \frac{6}{s^2 + 4} = F(s) = L\{\dots 3 \sin 2t \dots\}$$

$$\therefore L^{-1}\left\{\frac{6e^{-2s}}{s^2 + 4}\right\} = 3u(t - 2) \cdot \sin 2(t - 2)$$

Because

$$L^{-1}\left\{\frac{6e^{-2s}}{s^2 + 4}\right\} = u(t - 2) \cdot f(t - 2)$$

where $f(t) = L^{-1}\left\{\frac{6}{s^2 + 4}\right\}$

Example 3

Determine $L^{-1} \left\{ \frac{s \cdot e^{-s}}{s^2 + 9} \right\}$.

the numerator contains e^{-s} which indicates $u(t - 1)$

$$\frac{s}{s^2 + 9} = F(s) = L\{\cos 3t\}$$

$$f(t) = \cos 3t \quad \therefore f(t - 1) = \cos 3(t - 1).$$

$$\therefore L^{-1} \left\{ \frac{s \cdot e^{-s}}{s^2 + 9} \right\} = u(t - 1) \cdot \cos 3(t - 1)$$

Exercise

Determine the inverse transforms of the following

$$(a) \frac{2e^{-5s}}{s^3}$$

$$(b) \frac{3e^{-2s}}{s^2 - 1}$$

$$(c) \frac{8e^{-4s}}{s^2 + 4}$$

$$(d) \frac{2s \cdot e^{-3s}}{s^2 - 16}$$

$$(e) \frac{5e^{-s}}{s}$$

$$(f) \frac{s \cdot e^{-s/2}}{s^2 + 2}$$

Results – all very straightforward.

(a) $u(t - 5) \cdot (t - 5)^2$

(b) $3u(t - 2) \cdot \sinh(t - 2)$

(c) $4u(t - 4) \cdot \sin 2(t - 4)$

(d) $2u(t - 3) \cdot \cosh 4(t - 3)$

(e) $5u(t - 1)$

(f) $u(t - 1/2) \cdot \cos \sqrt{2}(t - 1/2).$

The main points are

$$(a) \left. \begin{aligned} u(t-c) &= 0 & 0 < t < c \\ &= 1 & t \geq c \end{aligned} \right\}$$

$$(b) \left. \begin{aligned} L\{u(t-c)\} &= \frac{e^{-cs}}{s} \\ L\{u(t)\} &= \frac{1}{s} \end{aligned} \right\}$$

$$(c) L\{u(t-c) \cdot f(t-c)\} = e^{-cs} \cdot F(s) \quad \text{where } F(s) = L\{f(t)\}$$

$$(d) \text{ If } F(s) = L\{f(t)\}, \text{ then } e^{-cs} \cdot F(s) = L\{u(t-c)\} \cdot f(t-c)$$

Example 1

Determine the expression $f(t)$ for which

$$L\{f(t)\} = \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}$$

We take each term in turn and find its inverse transform

$$(a) \quad L^{-1}\left\{\frac{3}{s}\right\} = 3L^{-1}\left\{\frac{1}{s}\right\} = 3 \quad \text{i.e. } 3u(t)$$

$$(b) \quad L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1) \cdot 4(t-1)$$

$$(c) \quad L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = u(t-2) \cdot 5(t-2)$$

$$F(t) = 3u(t) - u(t-1) \cdot 4(t-1) + u(t-2) \cdot 5(t-2)$$

