Mencari Solusi PD Menggunakan Transformasi Laplace

Febrizal, MT

Review

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{a}{s}$	a	$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{1}{s+a}$	e^{-at}	$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{n!}{s^{n+1}}$	t^n	$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$	$\frac{s}{s^2-a^2}$	$\cosh at$

Review

- Soal-soal

Transformasi dari Turunan

- Misalkan f'(t) menyatakan turunan pertama dari f(t), f"(t) menyatakan turunan kedua dari f(t), dst
- Sehingga:

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t$$

Integrating by parts

$$L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\{-se^{-st}\}\,\mathrm{d}t$$

$$\therefore\ L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

- Dengan mengganti f(t) menjadi f'(t), diperoleh

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Because

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

so

$$L\{f''(t)\} = -f'(0) + sL\{f'(t)\}$$

$$= -f'(0) + s(-f(0) + sL\{f(t)\})$$

Writing $L\{f(t)\} = F(s)$ as usual, we have

$$L\{f(t)\} = F(s)$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

- Dengan cara yang sama, maka diperoleh

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'0) - f''(0)$$

- **Notas Alternatif**
  - Misalkan x = f(t), pada saat t = 0, kita tulis

$$x = x_0 \qquad \text{i.e. } f(0) = x_0$$

$$\frac{dx}{dt} = x_1 \qquad \text{i.e. } f'(0) = x_1$$

$$\frac{d^2x}{dt^2} = x_2 \qquad \text{i.e. } f''(0) = x_2 \text{ etc.}$$

$$\therefore \quad \frac{d^n x}{dt^n} = x_n \qquad \text{i.e. } f^n(0) = x_n$$

- Jika kita menuliskan transformasi laplace x dengan $\bar{x}$, maka;

$$\bar{x} = L\{x\} = L\{f(t)\} = F(s).$$

- Dan,

$$L\{x\} = \bar{x}$$
$$L\{\dot{x}\} = s\bar{x} - x_0$$
$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$
$$L\{\dddot{x}\} = s^3\bar{x} - s^2x_0 - sx_1 - x_2$$

- Dengan demikian, maka

$$L\{\ddddot{x}\} = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3$$

Solusi PD Orde I

- Contoh 1
  - Tentukanlah solusi dari PD $\frac{dx}{dt} - 2x = 4$ jika pada saat t = 0, x = 1
- Penyelesaian
  - Tuliskan kembali persamaan tsb dalam Transformasi Laplace,

$$L\{x\} = \bar{x}; \quad L\{\dot{x}\} = s\bar{x} - x_0; \quad L\{4\} = \frac{4}{s}$$

  - Sehingga persamaan menjadi,

$$(s\bar{x} - x_0) - 2\bar{x} = \frac{4}{s}$$

- Masukkan nilai awal $t = 0 \rightarrow x = 1$, berarti $x_0 = 1$, sehingga

$$\therefore\ s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$$

- Susun kembali persamaan diatas sehingga diperoleh persamaan dalam $\bar{x}$,

$$\bar{x} = \frac{s+4}{s(s-2)}$$

- Lakukan invers transformasi pada persamaan tsb.

$$\frac{s+4}{s(s-2)} \equiv \frac{A}{s} + \frac{B}{s-2} \qquad \therefore\ s + 4 = A(s-2) + Bs$$

(1) Put $(s-2) = 0$, i.e. $s = 2$ $\qquad \therefore\ 6 = B(2) \qquad \therefore\ B = 3$

(2) Put $s = 0$ $\qquad \therefore\ 4 = A(-2) \qquad \therefore\ A = -2$

$$\therefore\ \bar{x} = \frac{s+4}{s(s-2)} = \frac{3}{s-2} - \frac{2}{s}$$

$$x = L^{-1}\left\{\frac{s+4}{s(s-2)}\right\} = L^{-1}\left\{\frac{3}{s-2} - \frac{2}{s}\right\} = .3e^{2t} - 2\ \ ..$$

- Contoh 2
  - Selesaikan PD $\frac{dx}{dt} + 2x = 10e^{3t}$, jika diketahui pada t = 0 → x = 6
- Penyelesaian
  - Nyatakan persamaan diatas dalam Transformasi Laplace

$$(s\bar{x} - x_0) + 2\bar{x} = \frac{10}{s-3}$$

  - Masukkan nilai awal $x_0 = 6$

$$s\bar{x} - 6 + 2\bar{x} = \frac{10}{s-3}$$

  - Susun ulang persamaan untuk mendapatkan $\bar{x}$,

$$\bar{x} = \frac{6s-8}{(s+2)(s-3)}$$

  - Gunakan invers laplace untuk mendapatkan x

$$x = L^{-1}\left\{\frac{6s-8}{(s+2)(s-3)}\right\} = 4e^{-2t} + 2e^{3t}$$

- Contoh 3
  - Carilah solusi PD $\frac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$, jika diketahui t = 0 → x = 0
- Penyelesaian

$$\frac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$$

(a) $(s\bar{x} - x_0) - 4\bar{x} = \frac{2}{s-2} + \frac{1}{s-4}$

(b) $x_0 = 0 \quad \therefore \; s\bar{x} - 4\bar{x} = \frac{2}{s-2} + \frac{1}{s-4}$

(c) $\therefore \; \bar{x} = \frac{2}{(s-2)(s-4)} + \frac{1}{(s-4)^2}$

(d) $\frac{2}{(s-2)(s-4)} \equiv \frac{A}{s-2} + \frac{B}{s-4} \quad \therefore \; 2 = A(s-4) + B(s-2)$

Putting $(s-2) = 0$, i.e. $s = 2 \quad \therefore \; 2 = A(-2) \quad \therefore \; A = -1$

Putting $(s-4) = 0$, i.e. $s = 4 \quad \therefore \; 2 = B(2) \quad \therefore \; B = 1$

$\therefore \; \bar{x} = \frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}$

$\therefore \; x = e^{4t} - e^{2t} + te^{4t}$

Latihan

Solve the following equations by Laplace transforms.

(a) $\dfrac{dx}{dt} + 3x = e^{-2t}$ given that $x = 2$ when $t = 0$

(b) $3\dot{x} - 6x = \sin 2t$ given that $x = 1$ when $t = 0$

Solusi PD Orde II

Example 1

Solve the equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$, given that at $t = 0$, $x = 5$ and $\frac{dx}{dt} = 7$.

(a) We rewrite the equation in terms of its transforms, remembering that

$$L\{x\} = \bar{x}$$
$$L\{\dot{x}\} = s\bar{x} - x_0$$
$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$

The equation becomes ....

$$(s^2\bar{x} - sx_0 - x_1) - 3(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}$$

(b) Insert the initial conditions. In this case $x_0 = 5$ and $x_1 = 7$

$$\therefore \quad (s^2\bar{x} - 5s - 7) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

(c) Rearrange to obtain $\bar{x} = \ldots\ldots\ldots\ldots$

$$s^2\bar{x} - 5s - 7 - 3s\bar{x} + 15 + 2\bar{x} = \frac{2}{s-3}$$

$$(s^2 - 3s + 2)\bar{x} - 5s + 8 = \frac{2}{s-3}$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s-3} + 5s - 8 = \frac{2 + 5s^2 - 23s + 24}{s-3}$$

$$\therefore \quad \bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

(d) Now for partial fractions

$$\frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$A = 4; \quad B = 0; \quad C = 1$$

$$\therefore \ \bar{x} = \frac{4}{s-1} + \frac{1}{s-3}$$

$$x = 4e^{t} + e^{3t}$$

- Contoh 2
  - Tentukan solusi PD $\frac{d^2x}{dt^2} - 4x = 24\cos 2t$ jika diketahui pada t = 0 → x = 3 dan dx/dt = 4
- Penyelesaian
  - Nyatakan persamaan dalam transformasi laplace

$$(s^2\bar{x} - sx_0 - x_1) - 4\bar{x} = \frac{24s}{s^2+4}$$

  - Masukkan nilai kondisi awal; x_0 = 3 dan x_1 = 4

$$s^2\bar{x} - 3s - 4 - 4\bar{x} = \frac{24s}{s^2+4}$$

$$\therefore\ (s^2 - 4)\bar{x} = 3s + 4 + \frac{24s}{s^2+4}$$

$$= \frac{3s^3 + 4s^2 + 36s + 16}{s^2+4}$$

$$\bar{x} = \frac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)}$$

$$\boxed{\frac{3s^3+4s^2+36s+16}{(s^2+4)(s-2)(s+2)} \equiv \frac{As+B}{s^2+4}+\frac{C}{s-2}+\frac{D}{s+2}}$$

$$\therefore\ 3s^3+4s^2+36s+16 \equiv (As+B)(s-2)(s+2)+C(s^2+4)(s+2) + D(s^2+4)(s-2)$$

Putting $(s-2)=0$, i.e. $s=2$, gives $C=4$

Putting $(s+2)=0$, i.e. $s=-2$, gives $D=2$

Equating coefficients of s^3 and also the constant terms gives $A=-3$ and $B=0$.

$$\therefore\ \bar{x}=\frac{3s^3+4s^2+36s+16}{(s^2+4)(s-2)(s+2)}=\frac{4}{s-2}+\frac{2}{s+2}-\frac{3s}{s^2+4}$$

$$\boxed{x=4e^{2t}+2e^{-2t}-3\cos 2t}$$

Contoh 3

- Selesaikan PD $\ddot{x} + 5\dot{x} + 6x = 4t,$ jika diketahui pada t = 0 → x = 0 dan $\dot{x} = 0.$

Penyelesaian

- Seperti biasa, nyatakan persamaan dalam transformasi laplace

$$(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 6\bar{x} = \frac{4}{s^2}$$

$$x_0 = 0;\ x_1 = 0 \quad \therefore\ (s^2 + 5s + 6)\bar{x} = \frac{4}{s^2}$$

$$\therefore\ \bar{x} = \frac{4}{s^2(s+2)(s+3)}$$

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{s(s+2)(s+3)}\right\} = \frac{1}{s}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right\}$$

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{6}.\frac{1}{s} - \frac{2}{(s+2)} + \frac{4}{3}.\frac{1}{s+3}\right\}$$

$$\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}.\frac{1}{s(s+3)}$$

- Uraikan lagi pecahan suku kedua dan ketiga dengan cara pecahan parsial

$$\bar{x} = \frac{2}{3} \cdot \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+2} + \frac{4}{9} \cdot \frac{1}{s} - \frac{4}{9} \cdot \frac{1}{s+3}$$

- Sederhanakan persamaan diatas, sehingga didapat

$$\bar{x} = \frac{2}{3} \cdot \frac{1}{s^2} - \frac{5}{9} \cdot \frac{1}{s} + \frac{1}{s+2} - \frac{4}{9} \cdot \frac{1}{s+3}$$

- Lakukan transformasi invers, maka

$$x = \frac{2}{3}t - \frac{5}{9} + e^{-2t} - \frac{4}{9}e^{-3t}$$

Example 4

Solve $\ddot{x} - 2\dot{x} + 10x = e^{2t}$, given that at $t = 0$, $x = 0$ and $\dot{x} = 1$.

$$(s^2\bar{x} - sx_0 - x_1) - 2(s\bar{x} - x_0) + 10\bar{x} = \frac{1}{s-2}$$

$$x_0 = 0;\ x_1 = 1 \qquad \therefore\ s^2\bar{x} - 1 - 2s\bar{x} + 10\bar{x} = \frac{1}{s-2}$$

$$\therefore\ (s^2 - 2s + 10)\bar{x} = 1 + \frac{1}{s-2} = \frac{s-1}{s-2}$$

$$\therefore\ \bar{x} = \frac{s-1}{(s-2)(s^2 - 2s + 10)}$$

$$\frac{s-1}{(s-2)(s^2-2s+10)} \equiv \frac{A}{(s-2)} + \frac{Bs+C}{s^2-2s+10}$$

$$\therefore\ s - 1 = A(s^2 - 2s + 10) + (s-2)(Bs + C)$$

Put $(s-2)=0$, i.e. $s=2$ $\quad 1 = A(4-4+10) \quad \therefore\ A = \dfrac{1}{10}$

$[s^2] \quad 0 = A + B \quad \therefore\ B = -\dfrac{1}{10}$

[CT] $\quad -1 = 10A - 2C \quad \therefore\ 2C = 2 \quad \therefore\ C = 1$

$$\therefore\ \bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-10}{s^2-2s+10}\right\}$$

- $$\frac{s-10}{s^2-2s+10} = \frac{s-10}{(s-1)^2+9} = \frac{(s-1)-9}{(s-1)^2+9}.$$

$$\bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-1}{(s-1)^2+9} + \frac{9}{(s-1)^2+9}\right\}$$

$$\boxed{x = \frac{1}{10}\{e^{2t} - e^{t}\cos 3t + 3e^{t}\sin 3t\}}$$

Example 5

Solve $\ddot{x} + \dot{x} + x = e^{-t}$ given that at $t = 0$, $x = 0$ and $\dot{x} = 1$. We find the expression for $\bar{x}$ as before.

Because $(s^2\bar{x} - sx_0 - x_1) + (s\bar{x} - x_0) + \bar{x} = \dfrac{1}{s+1}$ where $x_0 = 0$ and $x_1 = 1$ so that

$$s^2\bar{x} - 1 + s\bar{x} + \bar{x} = \frac{1}{s+1}$$

therefore

$$\bar{x}(s^2 + s + 1) - 1 + \frac{1}{s+1} - \frac{s+2}{s+1}$$

giving

$$\bar{x} = \frac{s+2}{(s+1)(s^2+s+1)}$$

$$\bar{x} = \frac{s+2}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$$

so that

$$s+2 = A\left(s^2+s+1\right) + (Bs+C)(s+1)$$

Put $s+1=0$, that is $s=-1$ then

	$1 = A(1-1+1)$	so that $A = 1$
$[s^2]$	$0 = A + B$	so that $B = -1$
[CT]	$2 = A + C$	so that $C = 1$

Therefore

$$\bar{x} = \frac{1}{s+1} - \frac{s-1}{s^2+s+1}$$

$$\frac{s-1}{s^2+s+1} = \ldots\ldots\ldots\ldots$$

$$\frac{s-1}{s^2+s+1} = \frac{s-1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{s+\frac{1}{2}-\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\bar{x} = \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\boxed{x = e^{-t} - e^{-t/2}\cos\frac{\sqrt{3}t}{2} + \sqrt{3}e^{-t/2}\sin\frac{\sqrt{3}t}{2}}$$

Latihan

- Carilah solusi dari persamaan diferensial dibawah ini:

$\ddot{x} - 7\dot{x} + 12x = 2$ given that at $t = 0$, $x = 1$ and $\dot{x} = 5$

$\ddot{x} - 2\dot{x} + x = te^{t}$ given that at $t = 0$, $x = 1$ and $\dot{x} = 0$.