Transformasi Laplace

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Definisi Transformasi Laplace

- Transformasi Laplace dari fungsi f(t) dinotasikan dengan L{f(t)} dan didefinisikan sebagai integral semi takterbatas:

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t$$

- Parameter s diasumsikan bernilai positif dan cukup besar untuk memastikan bahwa integralnya konvergen.
- Nilai yang dimasukkan kedalam batas integral adalah nilai t, sehingga hasil dari integralnya dinyatakan dalam s. Sehingga:

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = F(s)$$

- Contoh 1
  - Carilah transformasi Laplace dari f(t) = a (konst)
- Penyelesaian

$$L\{a\} = \int_0^\infty a e^{-st} \mathrm{d}t = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{a}{s}\{0 - 1\} = \frac{a}{s}$$

$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0)$$

- Contoh 2
  - Carilah transformasi Laplace dari f(t) = e^{at}
  - a adalah konstanta
- Penyelesaian

$$L\{e^{at}\} = \int_0^{\infty} e^{at} e^{-st}\, \mathrm{d}t = \int_0^{\infty} e^{-(s-a)t}\, \mathrm{d}t = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= -\frac{1}{s-a}\{0-1\} = \frac{1}{s-a}$$

$$\therefore\ L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a)$$

- Sekarang kita mempunyai dua buah transformasi standar, yaitu : $L\{a\} = \frac{a}{s}$ and $L\{e^{at}\} = \frac{1}{s-a}$

- Dengan demikian, tentukanlah:

$$\therefore \quad L\{4\} = \ldots\ldots\ldots\ldots; \qquad L\{e^{4t}\} = \ldots\ldots\ldots\ldots$$
$$L\{-5\} = \ldots\ldots\ldots\ldots; \qquad L\{e^{-2t}\} = \ldots\ldots\ldots\ldots$$

- Jawaban:

$$L\{4\} = \frac{4}{s}; \qquad L\{e^{4t}\} = \frac{1}{s-4}$$
$$L\{-5\} = -\frac{5}{s}; \qquad L\{e^{-2t}\} = \frac{1}{s+2}$$

- Dengan cara yang sama, diperoleh transformasi Laplace untuk fungsi berikut:

$$\therefore \; L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\therefore \; L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Sifat Transformasi Laplace

- Laplace adalah operator linier, sehingga;
  - Transformasi laplace dari fungsi penjumlahan, sama dengan jumlah dari transformasi laplace dari masing-masing fungsinya.

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

  - Transformasi laplace dari fungsi yang dikali dengan suatu konstanta, sama dengan konstanta dikali dengan transformasi laplace dari fungsi.

$$L\{kf(t)\} = kL\{f(t)\}$$

- Contoh

(a) $L\{2e^{-t} + t\} = L\{2e^{-t}\} + L\{t\}$

$$= 2L\{e^{-t}\} + L\{t\}$$

$$= \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2 + s + 1}{s^2(s+1)}$$

(b) $L\{2\sin 3t + \cos 3t\} = 2L\{\sin 3t\} + L\{\cos 3t\}$

$$= 2.\frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9}$$

(c) $L\{4e^{2t} + 3\cosh 4t\} = 4L\{e^{2t}\} + 3L\{\cosh 4t\}$

$$= 4.\frac{1}{s-2} + 3.\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16}$$

$$= \frac{7s^2 - 6s - 64}{(s-2)(s^2-16)}$$

- Latihan
  - Tentukanlah:
    1. $L\{2\sin 3t + 4\sinh 3t\} = \ldots\ldots\ldots\ldots$
    2. $L\{5e^{4t} + \cosh 2t\} = \ldots\ldots\ldots\ldots$
    3. $L\{t^3 + 2t^2 - 4t + 1\} = \ldots\ldots\ldots\ldots$
  - Penyelesaian:
    1. $$L\{2\sin 3t + 4\sinh 3t\} = 2.\frac{3}{s^2+9} + 4.\frac{3}{s^2-9}$$
       $$= \frac{6}{s^2+9} + \frac{12}{s^2-9} = \frac{18(s^2+3)}{s^4-81}$$
    2. $$L\{5e^{4t} + \cosh 2t\} = \frac{5}{s-4} + \frac{s}{s^2-4} = \frac{6s^2-4s-20}{(s-4)(s^2-4)}$$
    3. $$L\{t^3 + 2t^2 - 4t + 1\} = \frac{3!}{s^4} + 2.\frac{2!}{s^3} - 4.\frac{1!}{s^2} + \frac{1}{s}$$
       $$= \frac{1}{s^4}\{s^3 - 4s^2 + 4s + 6\}$$

Teorema pada Transf. Laplace

– Teorema 1

- Jika $L\{f(t)\} = F(s)$, maka $L\{e^{-at} f(t)\} = F(s+a)$

– Contoh

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$$

– Contoh lain

$$L\{t^2\} = \frac{2}{s^3} \quad \therefore \; L\{t^2 e^{4t}\} = . \; \frac{2}{(s-4)^3} \; .$$

Latihan

- Tentukanlah

1. $L\{e^{-2t}\cosh 3t\}$
2. $L\{2e^{3t}\sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t}\cos t\}$
5. $L\{e^{3t}\sinh 2t\}$
6. $L\{t^3 e^{-4t}\}$

1. $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$ $\quad\therefore\ L\{e^{-2t}\cosh 3t\} = \dfrac{s+2}{(s+2)^2 - 9} = \dfrac{s+2}{s^2 + 4s - 5}$

2. $L\{\sin 3t\} = \dfrac{3}{s^2 + 9}$ $\quad\therefore\ L\{2e^{3t}\sin 3t\} = \dfrac{6}{(s-3)^2 + 9} = \dfrac{6}{s^2 - 6s + 18}$

3. $L\{4t\} = 4.\dfrac{1}{s^2}$ $\quad\therefore\ L\{4te^{-t}\} = \dfrac{4}{(s+1)^2}$

4. $L\{\cos t\} = \dfrac{s}{s^2 + 1}$ $\quad\therefore\ L\{e^{2t}\cos t\} = \dfrac{s-2}{(s-2)^2 + 1} = \dfrac{s-2}{s^2 - 4s + 5}$

5. $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$ $\quad\therefore\ L\{e^{3t}\sinh 2t\} = \dfrac{2}{(s-3)^2 - 4} = \dfrac{2}{s^2 - 6s + 5}$

6. $L\{t^3\} = \dfrac{3!}{s^4}$ $\quad\therefore\ L\{t^3e^{-4t}\} = \dfrac{6}{(s+4)^4}$

– Teorema 2

- Jika L{f(t)} = F(s), maka L{t.f(t)} = - F'(s)

– Contoh

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\therefore\ L\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

– Contoh lain

$$L\{t\cosh 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = -\frac{(s^2-9)-s(2s)}{(s^2-9)^2} = \frac{s^2+9}{(s^2-9)^2}$$

- Dan $L\{t^2\cosh 3t\}$ adalah:

$$L\{t^2\cosh 3t\} = L\{t(t\cosh 3t)\} = -\frac{d}{ds}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}$$

$$= \frac{2s(s^2+27)}{(s^2-9)^3}$$

- Latihan
  - Jika diketahui $L\{\sin 4t\} = \dfrac{4}{s^2 + 16}$
  - Tentukanlah

$L\{t \sin 4t\} = \ldots\ldots\ldots\ldots$ and $L\{t^2 \sin 4t\} = \ldots\ldots\ldots\ldots$

  - Jawaban

$$\frac{8s}{(s^2+16)^2}; \quad \frac{8(3s^2-16)}{(s^2+16)^3}$$

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

– Teorema 3

- Jika $L\{f(t)\} = F(s)$, maka $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$

- Tetapi dengan syarat $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ ada nilainya.

– Contoh

- Tentukan $L\left\{\frac{\sin at}{t}\right\}$

- Pertama kita harus menguji $\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\} = \left\{\frac{0}{0}\right\} = ?$

- Dengan aturan L' Hopital, diferensialkan bagian atas dan bawah secara terpisah dan substitusi t = 0

- Sehingga $\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\} = \lim_{t\to 0}\left\{\frac{a\cos at}{1}\right\} = a$

- Karena limit tersebut mempunyai nilai, berarti teorema 3 bisa diterapkan.
- Karena $L\{\sin at\} = \frac{a}{s^2 + a^2}$, maka

$$\begin{aligned} L\left\{\frac{\sin at}{t}\right\} &= \int_s^\infty \frac{a}{\sigma^2 + a^2}\,\mathrm{d}\sigma \\ &= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty \\ &= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) \\ &= \arctan\left(\frac{a}{s}\right) \end{aligned}$$

- Cat:

$$\arctan\left(\frac{a}{s}\right) + \arctan\left(\frac{s}{a}\right) = \frac{\pi}{2}$$

– Contoh lain

- Tentukan $L\left\{\frac{1-\cos 2t}{t}\right\}$
- Pertama kita uji limit;

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1-1}{0} = \frac{0}{0} = ? \quad \therefore \text{ Apply l'Hôpital's rule.}$$

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \lim_{t\to 0}\left\{\frac{2\sin 2t}{1}\right\} = \frac{0}{1} = 0 \quad \therefore \text{ limit exists.}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

- Dengan teorema 3

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right\}\mathrm{d}\sigma$$

$$= \left[\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty} = \frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$

$$\sigma\to\infty,\ \ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\to\ln 1 = 0$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right) = \ln\left(\frac{s^2}{s^2+4}\right)^{-1/2}$$

$$= \ln\sqrt{\frac{s^2+4}{s^2}}$$